Transmission Line Models of Wire Scattering

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Abstract

The report describes the estimation of the current induced in a horizontal wire lying on the ground due to a magnetic dipole located directly above it. A transmission line model is employed to provide insights into the physical mechanisms. A key and novel component of the theory is the estimation of the propagation constant and the characteristic impedance of the line in proximity to the ground.

The results are compared with those from a commercial numerical simulation program called "FEKO". It is found that the transmission line model provides results in satisfactory agreement with the simulation. Therefore the model is expected to be useful in optimizing equipment for the automatic detection of buried wires in general.

Executive Summary

The calculation of the current induced in a wire by an electric dipole or a current loop is not trivial when the wire is located close to the ground and when the excitation frequency is high so that the quasi-static approximation is not valid. These situations are typically encountered in practical applications, such as the automatic detection of wires and cables. Analytic methods can be replaced by numerical simulation but these may be time-consuming and expensive. The purpose of this work is to develop an approximate analytic model based on transmission line theory.

Fortunately, parts of the transmission line theory are directly amenable to analytic methods. For example, the theory of the electromagnetic field from an antenna located above a conducting half space has been thoroughly explored by Sommerfeld and can be implemented to find the electric field at the position of a wire near the ground. The current in the wire can be estimated from the electric field if the propagation constant and the characteristic impedance of the wire (as part of a transmission line) are known. The main part of the report is the calculation of these last two quantities for both bare and insulated wires.

A feature of this study is the rapid transition in the propagation constants as the wire height is varied from just beneath to just above the ground. This indicates that the wire response is very sensitive to height as well as the condition of the ground surface and its compaction.

The results from the transmission line model are then compared with those from a commercial numerical simulation called “FEKO”. Satisfactory agreement has been found over a limited range of ground parameters and it is expected that this will facilitate progress in the automatic detection of buried wires. Moreover, calculations can be implemented very rapidly and inexpensively.

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Introduction

There are two objectives to this report. The first is to develop a simple theory to estimate the current induced in a wire lying on the surface of the ground by a current loop above the ground. The second objective is to compare the predictions of this theory with numerical simulations [1] based on a commercial package called "FEKO". Agreement between the two approaches provides insights into the physics of the problem and these could be useful in the optimization of equipment designed for the detection of utility wires and command wires for Improvised Explosive Devices (IEDs). The theory is based on a transmission line model.

The theory for the electric fields from electric dipoles and current loops above a conducting half space (representing the ground) has been treated by Sommerfeld [2]. This is valid over a wide range of frequencies and is not restricted to the quasi-static approximation. The electric field in the neighborhood of the wire can be calculated analytically, though the formulae involve integration over Bessel functions. When wires are located near a conducting half space, currents are induced in this half space and the half space tends to act as the return conductor of a transmission line. If the distance from the ground to a horizontal wire running parallel to the surface of the half space (ground) is much smaller than the length of the wire (as will usually be the case), the current in the wire can easily be estimated. However, this requires the propagation characteristics of the wire.

When the wire is buried deeply or is far above the ground, the propagation constants and the characteristic impedance can be estimated by applying full-space, rather than half-space models and calculations are much simpler than for the half-space model. These calculations provide partial verification of the general half-space approach.

To find the propagation constants of a horizontal wire located just above or just beneath the ground surface, the electromagnetic field must be calculated in the neighborhood of the wire carrying the current. The field is affected by the dielectric properties of the ground and its proximity. Conducting ground complicates the calculations because of the currents of free charges that are induced in it.

Firstly some basic electromagnetic theory and then transmission line theory is summarized; a simple coaxial line and a twin wire line are analyzed. It is important to compare the theory to that of simpler situations for which there is confidence about the veracity. Therefore we treat explicitly the case of a wire embedded in a homogeneous infinite conducting dielectric. Finally the case of a wire in a half space is considered.

Basic Theory

Transmission line theory is based on Maxwell's equations for electromagnetic fields within dielectric media. When an electric field is applied to a typical insulating dielectric, the field creates polarization throughout its volume [2]. The applied electric field tends to displace bound charges and the effect is represented by a dipole moment per unit volume. If the electric field is uniform, the dipole moments will cancel except at the dielectric

surface and there will be a surface charge distribution. Both volume and surface effects are handled by introducing the electric displacement, **D**, which is related to the electric field, **E**, by the relative permittivity, ε, and the permittivity of free space, ε_0:

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} \quad . \tag{1}$$

When the dielectric is conducting, free charges are available; the electric field can create a current of free charges and this is given by:

$$\mathbf{J} = \sigma\mathbf{E} . \tag{2}$$

Here, **J** is the current density of free charges and σ is the conductivity.

Maxwell's equations are then:

$$\begin{gathered} \nabla\cdot\mathbf{D} = \rho; \quad \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}; \quad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J} \end{gathered} , \tag{3}$$

where ρ is the density of free charges, **H** is the magnetic field and **B** is the magnetic induction and, in a similar manner to (1), we have:

$$\mathbf{B} = \mu\mu_0\mathbf{H} \ , \tag{4}$$

where μ is the relative permeability and μ_0 is the permeability of free space. In the present context in which media are non-magnetic, we typically assume that the relative permeability is just one.

For applications involving oscillating signals, such as antenna theory or for the signals induced in buried wires, it is convenient to assume a time dependence of the form $e^{j\omega t}$. Then we can re-write the second line of Maxwell's equations as:

$$\begin{gathered} \nabla\times\mathbf{E} = -j\omega\mathbf{B} \\ \nabla\times\mathbf{B} = \mu_0(j\omega\varepsilon\varepsilon_0 + \sigma)\mathbf{E} \end{gathered} . \tag{5}$$

A further simplification can be made by combining the two terms on the right hand side of the second equation by introducing a complex relative permittivity, ε', that includes both the displacement current (associated with the first term) and the conduction current of free charges. This is achieved through the following:

$$\varepsilon' = \varepsilon - \frac{j\sigma}{\omega\varepsilon_0} \ . \tag{6}$$

If the net density of free charges is zero, so that the medium is neutral, by using various vector identities these relations can be combined to yield:

$$\begin{gathered} \nabla^2\mathbf{E} + \varepsilon'\varepsilon_0\mu_0\omega^2\mathbf{E} = \mathbf{0} \\ \nabla^2\mathbf{B} + \varepsilon'\varepsilon_0\mu_0\omega^2\mathbf{B} = \mathbf{0} \end{gathered} . \tag{7}$$

These differential equations indicate that electromagnetic waves propagate with a wave number, k, given by:

$$k = \omega\sqrt{\varepsilon'\varepsilon_0\mu_0} \ . \tag{8}$$

Because the effective relative permittivity is complex, the waves are damped.

Solutions of electromagnetic problems are often assisted by introducing the magnetic vector potential, **A**, such that:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{9}$$

Substituting this into Maxwell's equations yields:

$$\begin{aligned} &\nabla \cdot \mathbf{A} = -j\omega\varepsilon'\varepsilon_0\mu_0 V \\ &\mathbf{E} = -j\omega\mathbf{A} - \nabla V; \qquad . \\ &\nabla^2\mathbf{A} + \omega^2\varepsilon'\varepsilon_0\mu_0\mathbf{A} = -\mu_0\mathbf{J}_{Applied} \end{aligned} \tag{10}$$

where the first equation removes arbitrariness in the definition of **A** in such a manner as to introduce a scalar potential, V, which is recognized as the usual potential of electrostatics. An applied current density $\mathbf{J}_{Applied}$ has been added to the last equation to represent the current for example in an electric dipole antenna. The currents induced in the dielectric medium are included on the left hand side of this equation.

For antenna theory, it is customary to solve the last of (10) and then to determine the electric and magnetic fields from the remaining equations. The last equation is a wave equation and the usual procedure is to assume that current density on the right hand side is the conduction current density in the antenna itself. The solution can be expressed in terms of an integral involving the retarded current density over the antenna volume, τ:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_{Applied}]d\tau}{r} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_{Applied}e^{-jkr}}{r}d\tau \ , \tag{11}$$

where r is the distance from the current density element to the point in question [2] [3]. This result can be understood by comparison with the solution of the differential equation for the electric potential of a dipole. It implies that the magnetic vector potential from a current element, $\mathrm{I}_z ds$ along the z-axis, only has a z-component in the same direction as the current element and is of the form:

$$\mathbf{A} = \frac{\mu_0 \mathrm{I}_z ds}{4\pi}\frac{e^{-jkr}}{r}\hat{\mathbf{z}}. \tag{12}$$

There is no dependence on any variable other than r so that we can check that it is a solution of the differential equation for the magnetic vector potential in (10) by expressing the vector Laplacian in a spherical coordinate system. Where there are no applied currents we have:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\mathrm{A}_z + k^2\mathrm{A}_z = 0. \tag{13}$$

It is easily verified that (12) is a solution of (13).

To illustrate how the magnetic vector potential is related to the magnetic induction for the current element, we can employ Stokes' Law:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \Phi_M = \int_C \mathbf{A} \cdot d\mathbf{s}. \tag{14}$$

This shows that the magnetic flux through a surface is equal to the line integral of the vector potential around it. If we consider rectangular loop with two sides parallel to the wire, there will be a contribution to the line integral from these sides but none from the remaining two sides as shown in Figure 1. Therefore, in the quasi-static limit ($k \to 0$), we have:

$$\begin{aligned}\Phi_M &= \mathrm{B}dxdz = \frac{\mu_0 \mathrm{I}ds}{4\pi}\left(-\frac{1}{r+dr}+\frac{1}{r}\right)dz \\ &= -\frac{\mu_0 \mathrm{I}ds}{4\pi}\frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{dx}dxdz = \frac{\mu_0 \mathrm{I}ds}{4\pi r^2}dxdz\sin\theta\end{aligned}, \tag{15}$$

where $x = r\sin\theta$ and the right hand rule has been used to determine the signs. Thus the familiar magneto-static Biot-Savart Law is recovered, with the induction in Figure 1 directed into the page:

$$\mathrm{B} = \frac{\mu_0 \mathrm{I}ds}{4\pi r^2}\sin\theta\,.$$

In the following, it is understood that the relative permittivity is complex so that from now onwards the prime on ε is omitted.

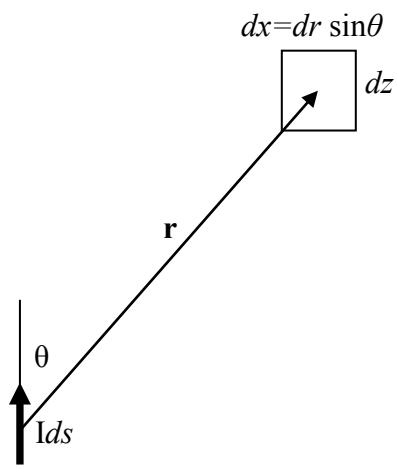

Figure 1. Geometry for the magnetic field from a current element. The loop and the vector **r** are in the plane of the paper.

Propagation Parameter Theory

The wave number and the characteristic impedance on a transmission line can be derived using an infinitesimal lumped circuit approximation. While many wave modes may be possible, we focus on the lowest order mode, which is typically a TEM mode. This approximation is equivalent to assuming that the propagation wavelength is large compared with the transverse dimensions of the line. This implies that the wave number is small compared to the reciprocal of these dimensions.

The method [3] requires knowledge of the inductance per unit length, L, and the capacitance per unit length, C. The impedance per unit length is then $Z = j\omega L$ and the admittance per unit length is $Y = j\omega C$. The wave number, k, and the characteristic impedance, Z_0, of the line are given by:

$$k = \sqrt{-ZY} = \omega\sqrt{LC}$$
$$Z_0 = \sqrt{Z/Y} = \sqrt{L/C} \quad . \tag{16}$$

For a long coaxial line filled with a dielectric with inner radius, *a,* and outer radius, *b*, the inductance per unit length is readily determined using Ampere's Law. The capacitance is obtained using Gauss's Law. The results are:

$$L = \frac{\mu\mu_0}{2\pi}\ln(b/a)$$
$$C = \frac{2\pi\varepsilon\varepsilon_0}{\ln(b/a)} \quad . \tag{17}$$

Therefore we have:

$$k = \omega\sqrt{\varepsilon\varepsilon_0\mu\mu_0}$$
$$Z_0 = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}} \quad . \tag{18}$$

As expected, the first of these is identical to (8) and the square root factor is the reciprocal of the phase velocity, *c*, of the waves on the line:

$$\frac{\omega}{k} = c = \frac{1}{\sqrt{\varepsilon\varepsilon_0\mu\mu_0}} \quad . \tag{19}$$

In a vacuum, this is just the speed of light.

The inductance and capacitance of the line are related directly to stored energy in magnetic and electric fields respectively. During wave propagation, this energy is transferred back and forth from inductance to capacitance. Because most of the electromagnetic energy is stored close to the wire, where the fields are greatest, the parameters are not sensitive to the outer radius of the coaxial line; this is clear from the logarithmic factor in the characteristic impedance.

The Twin Wire Line

The twin wire transmission line is relevant because the half space theory introduces an image of the wire in the ground just as if the ground were infinitely conducting. (There is another term that represents the remaining effects of the ground.) This image can be regarded as an additional conductor carrying a return current that forms the line. We assume that the centre of the actual wire is a height *h* above the ground so that the distance between the two centres is 2*h*. The wire and its image each have a radius of *a*.

The same approach as for the coaxial line can be employed but there is a significant complication. The electric field created by the second wire affects the charge distribution on the first; this causes an asymmetry in the charge distribution that increases as the wires approach each other. The effect can be handled by considering images in cylindrical conductors, by conformal transformation [3] or by the use of bipolar coordinates. Taking account of this effect for a twin line, the mutual capacitance per unit length is:

$$C = \frac{\pi\varepsilon\varepsilon_0}{\cosh^{-1}(h/a)} = \frac{\pi\varepsilon\varepsilon_0}{\log\left(\frac{h}{a} + \sqrt{\frac{h^2}{a^2} - 1}\right)} \quad . \tag{20}$$

When the wires are very close and h approaches a, the capacitance grows rapidly.

On the other hand, if the polarization effects are ignored, the charges on each wire can be considered to be at their respective centres and the electric field along a line joining the wire centres (between the wires) is given by:

$$E = -\frac{q}{2\pi\varepsilon\varepsilon_0(h+r)} - \frac{q}{2\pi\varepsilon\varepsilon_0(h-r)}, \tag{21}$$

where q is the charge per unit length. The electric potential difference between the wires is found by integrating with respect to r from $-h + a$ to $h - a$:

$$V = \frac{q}{2\pi\varepsilon\varepsilon_0}\log\left(\frac{h+r}{h-r}\right)_{-h+a}^{h-a} = \frac{q}{\pi\varepsilon\varepsilon_0}\log\left(\frac{2h}{a} - 1\right) \quad . \tag{22}$$

This yields a mutual capacitance per unit length:

$$C = \frac{\pi\varepsilon\varepsilon_0}{\log\left(\frac{2h}{a} - 1\right)} . \tag{23}$$

When $h >> a$, the approximate approach gives a satisfactory answer but, when the wires are close, errors could be significant.

The Full-Space Dielectric

When a wire is embedded in a conducting medium, typically there are two basic effects. Firstly, the permittivity is altered and this affects the capacitance per unit length. Secondly, electromotive forces are induced in the medium by the changing magnetic field and these create currents in the medium. These eddy currents create a back-electromotive force. Therefore the inductance per unit length is changed as well. The conductance can also be regarded as contributing an imaginary part to the permittivity, which changes the electric fields and creates losses. However, the induced currents in the medium must be handled from first principles using Maxwell's equations. As noted previously, the results can be expressed by differential equations and, in the medium where there are no applied currents; we have for example [3]:

$$\begin{aligned} \nabla^2\mathbf{J} &= -\omega^2\mu\mu_0\varepsilon\varepsilon_0\mathbf{J} \\ \nabla^2\mathbf{E} &= -\omega^2\mu\mu_0\varepsilon\varepsilon_0\mathbf{E} \quad . \\ \nabla^2\mathbf{H} &= -\omega^2\mu\mu_0\varepsilon\varepsilon_0\mathbf{H} \end{aligned} \tag{24}$$

For this application, it is convenient to employ a cylindrical coordinate system (r, θ, z) (rather than a spherical system), where the wire carrying the current lies along the z-axis. Again there is only one component of the magnetic vector potential. The differential equation for the principal variables can be solved using the method of separation of variables. In this case there is azimuthal symmetry and also we assume that the

propagation wavelength is large. Therefore only variation in the radial direction is important; variations along the direction of the wire are ignored.

In this case the components of the vector Laplacian in cylindrical coordinates (for some vector **v**) are:

$$\nabla^2 \mathbf{v} = \begin{vmatrix} \dfrac{\partial^2 \mathrm{v}_r}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \mathrm{v}_r}{\partial r} - \dfrac{\mathrm{v}_r}{r^2} \\ \dfrac{\partial^2 \mathrm{v}_\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \mathrm{v}_\theta}{\partial r} - \dfrac{\mathrm{v}_\theta}{r^2} \\ \dfrac{\partial^2 \mathrm{v}_z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \mathrm{v}_z}{\partial r} \end{vmatrix} . \tag{25}$$

The magnetic field has only one non-zero component, which is in the azimuthal direction. From (24) and (25), it is given by a solution of Bessel's equation [4]:

$$\frac{\partial^2 \mathrm{H}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{H}_\theta}{\partial r} + \left(\omega^2 \mu\mu_0 \varepsilon\varepsilon_0 - \frac{1}{r^2}\right)\mathrm{H}_\theta = 0 . \tag{26}$$

The solution that is appropriate corresponds to waves propagating outwards from the wire [4]; these are the Hankel functions, $H_1^{(2)}$ of order one of the second type (see Appendix B). The result is:

$$\mathrm{H}_\theta = \frac{\mathrm{I}}{2\pi a}\frac{H_1^{(2)}(kr)}{H_1^{(2)}(ka)} . \tag{27}$$

Here, we have used Ampere's Law to find the magnetic field at the wire surface and to determine the appropriate coefficient. Again the variable k is the wave number of waves propagating in the medium.

Implicit in this derivation is that the magnetic field is zero at infinity and therefore the net current (including displacement current) passing through a large circle around the wire is also close to zero. In turn this implies that the medium carries a return current so that the arrangement resembles a type of coaxial transmission line.

The inductance per unit length is given by:

$$L = \frac{\mu\mu_0}{\mathrm{I}}\int_a^\infty \mathrm{H}_\theta dr = \frac{\mu\mu_0}{2\pi a k}\frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)} \tag{28}$$

The integral in (28) has been evaluated using the relation [4]: $H_0'(z) = -H_1(z)$.

We emphasize that the expression for the inductance is expected to be valid only when $ka \ll 1$, because variations along the direction of the wire have been neglected.

The electric field has components in both the axial and radial directions but the component in the axial direction does not contribute to the capacitance of the line. The differential equation for the radial component is:

$$\frac{\partial^2 \mathrm{E}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{E}_r}{\partial r} + \left(\omega^2 \mu\mu_0 \varepsilon\varepsilon_0 - \frac{1}{r^2}\right)\mathrm{E}_r = 0 \ . \tag{29}$$

Its solution is:

$$\mathrm{E}_r = \frac{q}{2\pi\varepsilon\varepsilon_0 a}\frac{H_1^{(2)}(kr)}{H_1^{(2)}(ka)} . \tag{30}$$

where q is the charge per unit length on the wire and Gauss's Law has been employed to determine the constant. The capacitance per unit length is found from:

$$C = \frac{q}{-\int_b^\infty \mathrm{E}_r dr} = 2\pi\varepsilon\varepsilon_0 ak \frac{H_1^{(2)}(ka)}{H_0^{(2)}(ka)} \ . \tag{31}$$

Therefore, using (16), we find that the propagation parameters are:

$$\begin{aligned} k &= \omega\sqrt{\varepsilon\varepsilon_0 \mu\mu_0} \\ Z_0 &= \frac{1}{2\pi ak}\frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)}\sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}} \ . \end{aligned} \tag{32}$$

We note that the complex speed (which includes attenuation) of the waves along the line is identical to the complex speed of waves propagating freely in the medium.

Again we emphasize that these results are only valid when the transverse dimensions of the line are small compared to the wavelength of waves propagating on the line.

The Half-Space Dielectric

In practice we need the propagation constants for a horizontal wire in a half-space corresponding to a flat earth. The wire may be on or slightly above the surface or it might be buried at a shallow depth. It is important to note that waves cannot propagate along a single isolated wire in free space and some other conductor must be present if propagation is to occur. For example an antenna requires two elements, which must be conducting, though one of these can be the ground.

The presence of a conducting ground makes a fundamental difference to the theory. This can be understood by considering lines of electric displacement. These can only end on free charges. For example, if the ground is completely insulating, electric displacement lines emanating from the wire encounter no free charges and must end at infinity. In contrast, even slight ground conductivity implies that all electric displacement lines will eventually end in the ground.

If the wire is well above the surface by a distance of many wire radii, the fields near to the wire will not be significantly affected by the ground and the ground may be replaced by a perfectly conducting sheet. The model then comprises the wire and an image below ground level carrying an opposite return current. The wave number is just that in free space and the characteristic impedance of the line is (half) that of a parallel wire transmission line in air.

Similarly, if the wire is well below the ground surface, the wave number is that for waves propagating in an infinite ground and the characteristic impedance is analogous to a coaxial wire transmission line.

Practical cases tend to lie between these extremes. An analysis is important because an error in the wave number and the characteristic impedance can have a great influence on the predicted current induced in the wire by an external source.

The theory for the infinite medium is simplified by its symmetry and it is reasonable to suppose that the case of a horizontal wire in a half space might be treated in a similar manner. However, it seems to be difficult to solve the problem directly with this approach. Therefore, we build on the original work of Sommerfeld 0 in which the fields are found for a horizontal elementary current element, for example as part of a practical electric dipole.

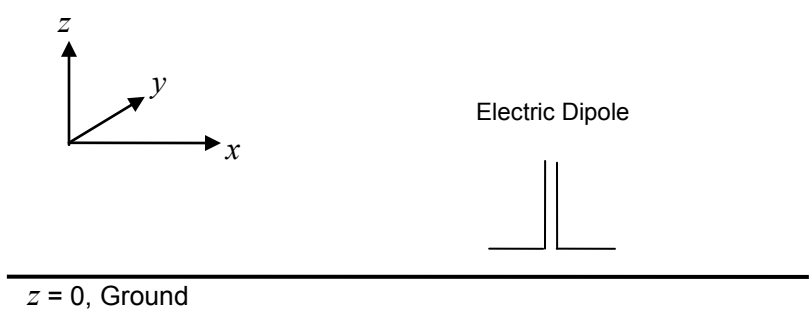

Figure 2. The configuration of axes, the ground and an electric dipole. The electric dipole is parallel to the wire and the x-axis.

Sommerfeld considers a short current element (electric dipole) located at a distance $z = h$ above a conducting half space. The origin of the coordinate system lies at ground level. The geometry is shown in Figure 2. The theory could be based on **A** but Sommerfeld employs the Hertz vector, **Π**, which is proportional to it. For consistency with Sommerfeld we also use the Hertz potential, though some care is needed to obtain the correct coefficients. For a vertical current element the representation requires a single component Π_z but for a horizontal element Sommerfeld shows that both horizontal and vertical components are needed to satisfy the boundary conditions at the ground. We assume that the current element is horizontal and parallel to the x-direction so that:

$$\mathbf{\Pi} = (\Pi_x, 0, \Pi_z)\,. \tag{33}$$

The electric and magnetic fields are determined from the following relations due to Sommerfeld 0; k is replaced by k_E in the lower medium:

$$\mathbf{E} = k^2 \mathbf{\Pi} + \nabla(\nabla \cdot \mathbf{\Pi})$$
$$\mathbf{H} = \frac{k^2}{j\omega\mu_0} \nabla \times \mathbf{\Pi} \quad . \tag{34}$$

Comparing these equations with (9) and (10) and taking into account that Sommerfeld employs a time dependence $e^{-j\omega t}$ rather than our $e^{j\omega t}$, the relation between Sommerfeld's Hertz potential and the magnetic vector potential is:

$$\mathbf{\Pi}^* = -\frac{j\omega}{k^2}\mathbf{A} \ . \tag{35}$$

Here the left hand side is a complex conjugate.

For example, from (34) and apart from constants of proportionality, some field components are given by:

$$\mathrm{E}_z = k^2 \Pi_z + \frac{\partial}{\partial z}\left(\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_z}{\partial z}\right)$$
$$\mathrm{H}_y = \frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x} \quad . \tag{36}$$

After applying the boundary conditions (the transverse components of **E** and **H** are continuous across the air-ground interface), the Hertz potentials (apart from constants of proportionality) are given by Sommerfeld (0 pages 259 to 261) for $z > 0$ as:

$$\Pi_x = \frac{e^{jkR}}{R} - \frac{e^{jkR'}}{R'} + 2\int_0^\infty J_0(\lambda r) e^{-u(z+h)} \frac{\lambda d\lambda}{u + u_E}$$
$$\Pi_{xE} = \frac{2}{n^2}\int_0^\infty J_0(\lambda r) e^{u_E z - uh} \frac{\lambda d\lambda}{u + u_E}$$
$$\Pi_z = -\frac{2}{k^2}\cos\varphi \int_0^\infty J_1(\lambda r) e^{-\mu(z+h)} \frac{u - u_E}{n^2 u + u_E} \lambda^2 d\lambda \quad . \tag{37}$$
$$\Pi_{zE} = -\frac{2}{k_E^2}\cos\varphi \int_0^\infty J_1(\lambda r) e^{u_E z - uh} \frac{u - u_E}{n^2 u + u_E} \lambda^2 d\lambda$$

Here, the subscript E refers to the ground, r is the horizontal distance of a point from the origin (cylindrical polar coordinate), $R = ((z - h)^2 + r^2)^{1/2}$ is the actual distance from the origin and R' is the distance of a point from an image in the ground, i.e. $R' = ((z + h)^2 + r^2)^{1/2}$. The complex refractive index of the ground is $n = k_E/k$ and J_0 and J_1 are Bessel functions of the first kind. The remaining variables are given by:

$$u = \sqrt{\lambda^2 - k^2}$$
$$u_E = \sqrt{\lambda^2 - k_E^2} \ . \tag{38}$$

and k and k_E are the wave numbers for air and ground respectively. As noted, these can be found directly from the complex permittivities of the two media. The angle φ is the azimuthal coordinate around the z-axis.

We shall also need the following identities:

$$\frac{e^{jkR}}{R} = \int_0^\infty J_0(\lambda r) e^{-u(z-h)} \frac{\lambda d\lambda}{u}, \quad z > h$$
$$\frac{e^{jkR'}}{R'} = \int_0^\infty J_0(\lambda r) e^{-u(z+h)} \frac{\lambda d\lambda}{u}, \quad z > h \qquad (39)$$

The first of (37) indicates that the x-component of the Hertz potential is the sum of a potential from a current element in the absence of the ground, that from an image located beneath the ground (as if it were infinitely conducting) and a correction term.

Half Space Propagation Constants

To find the fields from a long wire carrying a constant current, it is necessary to integrate the field contributions from all the current elements along the wire, which lies parallel to the x-axis at height, h. Because field variations along the x-axis can be neglected as before ($\partial/\partial x = 0$), we only need the fields on the z-axis at $x = 0$ and $y = 0$. If we set $y = 0$, then $r = x$. Then we can integrate over x, which occurs only in those terms containing a Bessel function. Integrating a Bessel function, $J_0(\lambda x)$ from 0 to infinity gives λ^{-1} [4]. However, we need to integrate over the entire wire and so the result must be doubled. We have also to integrate the first terms in Π_x; this is accomplished using the identities in (39).

To illustrate the procedure consider a wire lying above the interface. To find the inductance per unit length we need the magnetic induction at each point along the z-axis either above the wire out to infinity or below the wire down to an infinite depth. (In this analysis we neglect the internal inductance of the wire.) For simplicity we choose the upper space.

We deal with the two components of the Hertz potential separately. For the x-component we need the first equation in (37) with the identities in (39), i.e.

$$\begin{aligned}\Pi_x &= \frac{e^{jkR}}{R} - \frac{e^{jkR'}}{R'} + 2\int_0^\infty J_0(\lambda r) e^{-u(z+h)} \frac{\lambda d\lambda}{u + u_E} \\ &= \int_0^\infty J_0(\lambda r)\lambda d\lambda \left(\frac{e^{-u(z-h)} - e^{-u(z+h)}}{u} + 2\frac{e^{-u(z+h)}}{u + u_E} \right). \\ &= \int_0^\infty J_0(\lambda r)\lambda d\lambda \left(\frac{e^{-u(z-h)}}{u} + \frac{u - u_E}{u(u + u_E)} e^{-u(z+h)} \right)\end{aligned} \qquad (40)$$

Integrating over x and inserting the correct coefficient by comparison of (40) with (12) yields:

$$\Pi_x = \frac{\mu_0 \mathrm{I}}{2\pi} \int_0^\infty d\lambda \left(\frac{e^{-u(z-h)}}{u} + \frac{u - u_E}{u(u + u_E)} e^{-u(z+h)} \right). \qquad (41)$$

On the z-axis, we have (disregarding the overall sign)

:

$$\mathrm{B}_y(z) = \frac{\partial \Pi_x}{\partial z} = \frac{\mu_0 \mathrm{I}}{2\pi} \int_0^\infty d\lambda \left(e^{-u(z-h)} + \frac{u - u_E}{u + u_E} e^{-u(z+h)} \right) \tag{42}$$

When $u = u_E$, the entire medium is uniform and the second term in the integral is zero. In this case and in the limit $k \to 0$, the field is given by Ampere's Law, as expected:

$$\mathrm{B}_y = \frac{\mu_0 \mathrm{I}}{2\pi(z-h)} \tag{43}$$

However, there is a potential problem because the theory has not fully accounted for the wire itself, which is not infinitely thin and displaces the dielectric throughout its cross-section. Nevertheless, we can force the correct boundary condition at the wire surface as for the case of the wire embedded in an infinite medium and write:

$$\mathrm{B}_y = \frac{\mu_0 \mathrm{I}}{2\pi a} \frac{\int_0^\infty e^{-u(z-h)} d\lambda}{\int_0^\infty e^{-ua} d\lambda} \ , \quad z - h \geq a \ . \tag{44}$$

It should be no surprise that the integrals are just proportional to Hankel functions and the result is identical to (27). Some pertinent details are provided in Appendix B. This type of correction should be employed if the argument of the Hankel function is not sufficiently small to warrant the small argument approximation. Then we have:

$$\mathrm{B}_y = -j \frac{\mu_0 \mathrm{I}}{\pi^2 ka} \frac{\int_0^\infty e^{-u(z-h)} d\lambda}{H_1^{(2)}(ka)} = \frac{\mu_0 \mathrm{I}}{2\pi a} \frac{H_1^{(2)}(kr)}{H_1^{(2)}(ka)} \ . \tag{45}$$

(Because we have reverted to a time dependence $e^{j\omega t}$, Hankel functions of type 2 are appropriate as before.) It is worth noting that, to preserve the boundary conditions at the ground surface, the same normalization factor must be applied to all terms in the expressions for the magnetic field and the Hertz vector.

In the general case (but omitting the normalization), the flux per unit length of the wire is found by integrating over z from $h + a$ to ∞ and this is:

$$\begin{aligned} \Phi_M &= \Pi_x(\infty) - \Pi_x(h+a) \\ &= \frac{\mu_0 \mathrm{I}}{2\pi} \int_0^\infty d\lambda \left(\frac{e^{-ua}}{u} + \frac{u - u_E}{u(u+u_E)} e^{-u(2h+a)} \right) \\ &= \frac{\mu_0 \mathrm{I}}{2\pi} \int_0^\infty d\lambda \left(\frac{e^{-ua}}{u} - \frac{e^{-u(2h+a)}}{u} + \frac{2}{u+u_E} e^{-u(2h+a)} \right) \end{aligned} \ . \tag{46}$$

We see that the first two terms correspond to fields from the wire and its image in the ground plus a correction term. When the dielectric media are the same, this correction term cancels the image and we are left with only the field from the wire.

There is no contribution from the z-component of the Hertz potential because there is no variation in the x-direction. Therefore, the inductance per unit length for $h > 0$ is given by:

$$L = \frac{\mu_0}{2\pi}\int_0^\infty d\lambda\left(\frac{e^{-ua}}{u} - \frac{e^{-u(2h+a)}}{u} + \frac{2}{u+u_E}e^{-u(2h+a)}\right). \tag{47}$$

The first two terms represent the magnetic field from the wire and its image in free space and this inductance component is purely real. For a conducting earth, the last term is complex. A numerical analysis confirms that it is not valid when the dimensions of the line ($2h$) are not much less than the wavelength of waves propagating on the line. Thus there is a minor problem when the wire is at an extremely large height above the ground. The inductance does not necessarily go over properly to the expected values; sometimes a small negative resistance can occur. Nevertheless, this does not represent a practical problem as under these circumstances the ground can be regarded as infinitely conducting.

The capacitance of the line can be derived from the fields due to a stationary point charge; the charge varies sinusoidally in time. The technique is verified for a simple case where the dielectric is non-conducting in Appendix A. The result in general for $h > 0$ is:

$$C = \frac{2\pi\varepsilon_0}{\int_0^\infty d\lambda\left(\frac{e^{-ua}}{u} - \frac{e^{-u(2h+a)}}{u} + \frac{2}{u+\varepsilon_E u_E}e^{-u(2h+a)}\right)}. \tag{48}$$

For emphasis, a subscript E has been added to the relative permittivity of the ground.

The integrals in (47) and (48) closely resemble each other but are not identical because of the appearance of the relative permittivity of the ground in the capacitance. The form of the last term in the integral of (48) suggests that the ground often behaves almost as a perfect conductor as far as the electric field is concerned. When the wire is very close to the ground, the correction term may dominate. In short, both the wave number and the characteristic impedance of the line are affected.

When the wire is buried in the ground, the inductance and capacitance can be calculated by reversing the roles of the air and ground. This involves switching k and k_E as well as u and u_E, etc.

Numerical Integration

The first two terms in the integrations of (47) and (48) involve a pole at $u = 0$. In the model for a wire in air above the interface, the wave number is purely real so that the pole lies on the real axis. If the integration is implemented numerically, this presents two problems; the correct sign for a square root must be chosen and the integration path must be deformed to avoid the pole. In principle, a solution to both is to add a small negative imaginary part to the wave number to move it off the real axis. However, this is not satisfactory for the second problem because, to maintain accuracy, the imaginary part cannot be too large. If it is too small, the integrand must be evaluated many times near the

pole. Of course, the path of integration can also be moved off the real axis well away from the pole.

A better solution is to use the known analytic result for this type of integral, which involves Hankel functions. Moreover, when the argument of the Hankel function is very small, it can be replaced by a logarithm. In practice this occurs when the wire radius and the distance between the wire and its image in the ground are much less than the wavelength on the wire. In this case we have for the inductance per unit length:

$$L = \frac{\mu_0}{2\pi}\left(\log\left(\frac{2h-a}{a} \right) + \int_0^\infty d\lambda \frac{2}{u+u_E} e^{-u(2h+a)} \right) . \tag{49}$$

When $h >> a$, the logarithmic term represents a good approximation to half of the inductance of a twin wire transmission line, which is appropriate to a single wire above a perfect conductor. It should be noted that a negative sign is now in the logarithm. This is to obtain good functional behavior when the wire is close to the ground, though the result is unphysical when $|h| < a$; the wire and its image overlap. It may be possible to set the logarithm to zero leaving the correction integral term; in practice it may be necessary to add the self-inductance of the wire, which takes into account the magnetic field within the wire. However, this is usually negligible.

The integral exhibits a pole that occurs when:

$$2\lambda^2 = k^2 + k_E^2 . \tag{50}$$

This is typically dominated by the imaginary part of the ground wave number and lies well off the real axis. Therefore it is not so critical as long as the ground conductivity is significant; the integral can be evaluated readily but with some care.

As an example, the inductance per unit length and the capacitance per unit length are plotted in Figs. 3 and 4 for the parameters (last two derived) in Table 1. Both the real and imaginary parts of the inductance per unit length are almost constant over the range of wire centre heights that range from -10 cm to 10 cm. It turns out that the correction term closely compensates for any changes in the logarithmic term. The capacitance per unit length does change rapidly as the wire height passes through the air-ground interface. There is a minor hiatus here, which is associated with the difficulty of evaluating the integrals in this region.

The full-space inductance per unit length can be found from (28) for the parameters in Table 1 and is (1.85 – 0.22j) μH/m. Therefore it is clear that the inductance is dominated by the ground characteristics over a large range of heights above the ground. The logarithmic term only starts to dominate at heights of several metres.

The full-space capacitance per unit length is (17.38 -19.61j) pF/m. The plot in Fig. 4 is consistent with this but the air interface is affecting the imaginary part down to depths of meters.

The wave number and the characteristic impedance of a bare wire are plotted in Figs. 5 and 6. The wave number in the full-space medium is (0.194 - 0.102j) rad/m and the free-space wave number is 0.105 rad/m. These are consistent with the plot in Fig. 5. Once again the air-ground interface influences the wave number over metre heights and depths. The full-space characteristic impedance in the ground is (249.0 + 94.9j) Ω.

Table 1
Calculation Parameters

Parameter	
Wire radius (mm)	0.5
Ground Relative Permittivity	2.5
Ground Conductivity (S/m)	10^{-3}
Frequency (MHz)	5.0
Wave Number in Air (rad/m)	0.105
Wave Number in Ground (rad/m)	0.194 – j0.102

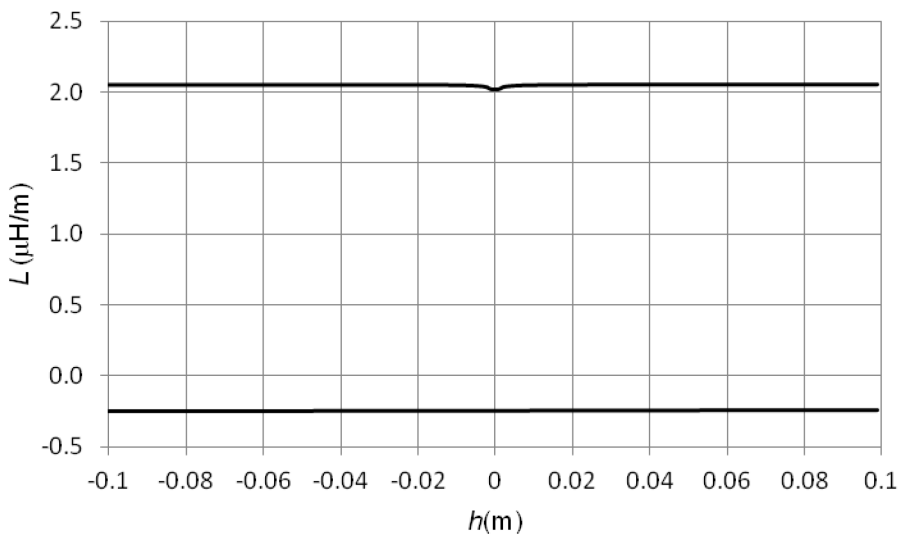

Figure 3. The real (upper) and imaginary (lower) parts of the inductance per unit length as a function of wire centre height, *h*, over ground.

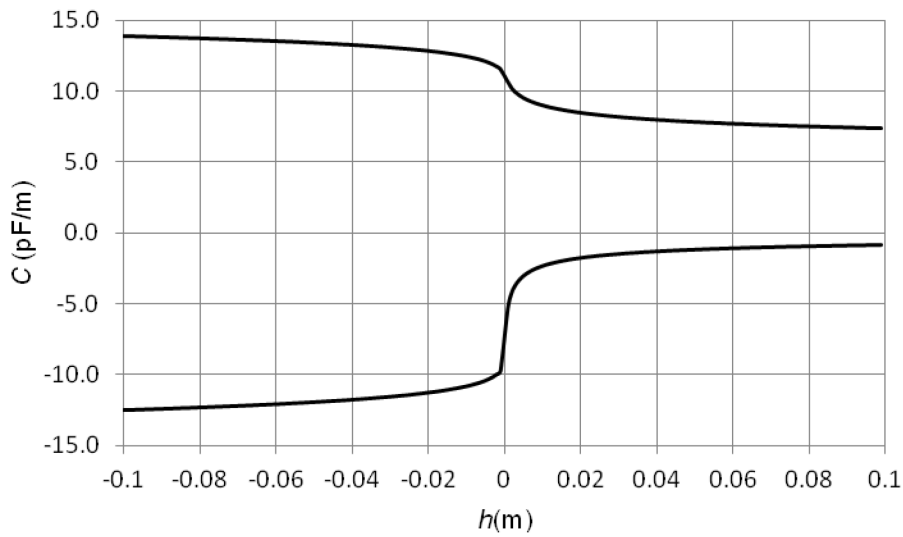

Figure 4. The real (upper) and imaginary (lower) parts of the capacitance per unit length as a function of wire centre height, *h*, over ground.

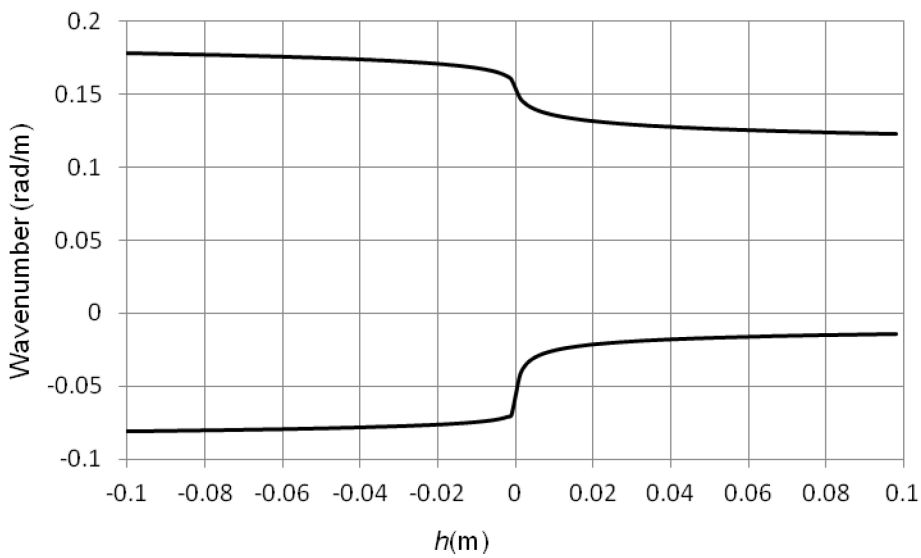

Figure 5. Real (upper) and imaginary (lower) parts of the wave number as a function of wire centre height, *h*, over ground.

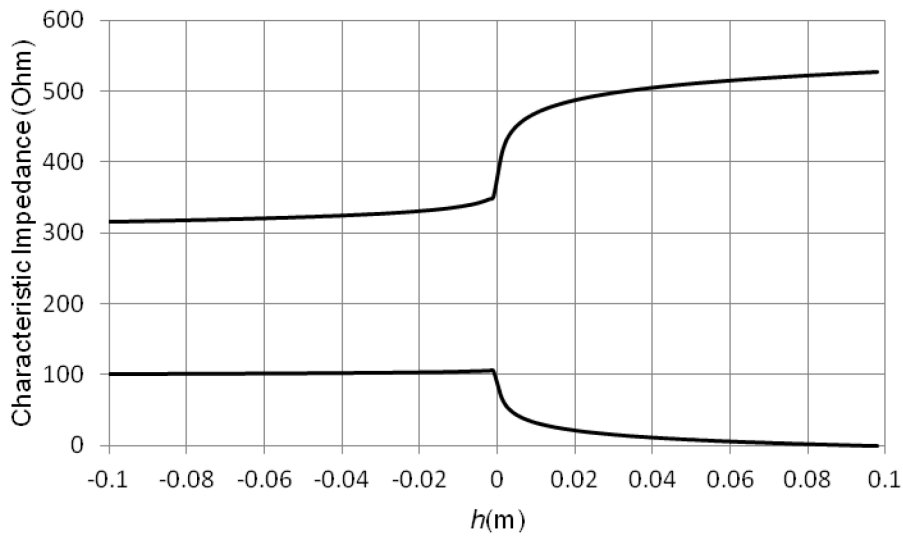

Figure 6. Real (upper) and Imaginary (lower) parts of the characteristic impedance as a function of wire centre height, *h*, over ground.

In summary it can be seen that the propagation parameters of a thin wire change very rapidly as the ground is approached from above or below; most of the changes occur within 2 cm of the ground. However, residual effects persist over much greater heights. At very large heights or depths the wave numbers go over to their full-space values. There is noticeable damping of waves on the wire at centimetric heights in air and considerable damping in the ground. The characteristic impedance is mainly resistive but typically has a significant inductive component in the ground and a slight capacitive component in air. Though the model has some problems when the wire is close to the ground, for the most part it is possible to interpolate and achieve reasonable results even when the wire is only partly buried.

The Field from a Magnetic Dipole

The field from a current loop is equivalent to a magnetic dipole. This type of duality is discussed in detail in [5]. The magnetic moment given by:

$$\mu_M = \mu_0 \mathrm{I} A \quad . \tag{51}$$

where I is the current and A is the area of the loop. The magnetic moment is often expressed without the permeability factor in units of A.m^2.

The equivalence is useful though free magnetic poles do not exist. This is because Maxwell's equations can easily be modified to include magnetic poles and this enables the behavior of a magnetic dipole to be analyzed using simple transformations. These involve replacing the electric field **E** by **H**, **H** by –**E** and reversing the roles of the permittivity and permeability. Thus the behavior of a magnetic dipole resembles that of the electric dipole but with azimuthally symmetric lines of electric field rather than magnetic field.

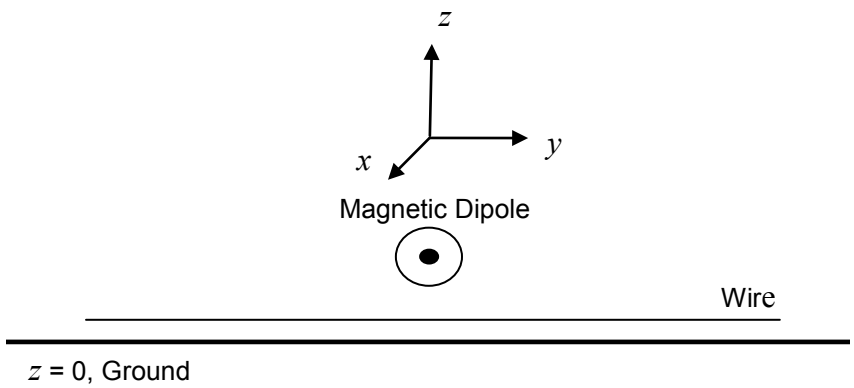

Figure 7. The configuration of axes, ground, wire and magnetic dipole.

The horizontal magnetic dipole corresponds to a current loop with its plane in the vertical direction, i.e. its normal is horizontal. The configuration is shown in Figure 7. The coordinate system differs from Figure 2 for consistency with Sommerfeld. We let the vertical direction be represented by the z-coordinate and the dipole lies in the horizontal x-direction, directly above the wire centre. Sommerfeld 0 provides the information to deduce the Hertz vectors for the horizontal magnetic dipole, which are of the same form as for the horizontal current element. However, in the present context there are some changes and the wire is at a height $z < h$, while previously we were interested in $z > h$; these affect Π_x, and, because the calculations to satisfy the boundary conditions resemble those for the vertical electric dipole, we can show that:

$$\begin{aligned}\Pi_x &= \int_0^\infty J_0(\lambda r)\lambda d\lambda \left(\frac{e^{u(z-h)} + e^{-u(z+h)}}{u} - \frac{2u_E}{u(n^2u + u_E)} e^{-u(z+h)} \right) \\ &= \int_0^\infty J_0(\lambda r)\lambda d\lambda \left(\frac{e^{u(z-h)}}{u} + \frac{n^2u - u_E}{u(n^2u + u_E)} e^{-u(z+h)} \right)\end{aligned} . \tag{52}$$

It can be verified that this is consistent with the first line of Sommerfeld's equation 32.10.

The electric and magnetic fields are now given by expressions of the form 0:

$$\begin{aligned}\mathbf{E} &= -\frac{k^2}{j\omega\varepsilon} \nabla \times \mathbf{\Pi} \\ \mathbf{H} &= k^2 \mathbf{\Pi} + \nabla(\nabla \cdot \mathbf{\Pi})\end{aligned} . \tag{53}$$

Comparison with FEKO

The FEKO simulations were conducted at various frequencies with a magnetic dipole exciter oriented horizontally perpendicularly to the wire and directly above it. We let the dipole be on the x-axis and the wire parallel to the y-axis (see Figure 7). We are interested

in the y-component of the electric field at $x = 0$ (disregarding constants of proportionality):

$$\mathrm{E}_y = \frac{\partial \Pi_z}{\partial x} - \frac{\partial \Pi_x}{\partial z}. \tag{54}$$

Therefore, since $\varphi = \pi/2$, $\cos\varphi = 0$ in Π_z, we have:

$$\mathrm{E}_y = -\frac{\partial \Pi_x}{\partial z} = -\int_0^\infty J_0(\lambda y)\lambda d\lambda \left(e^{u(z-h)} - \frac{n^2 u - u_E}{n^2 u + u_E} e^{-u(z+h)} \right). \tag{55}$$

To determine the correct constant of proportionality we let $n = 1$, $u = u_E$ and $y = z = 0$. Then, by differentiating formula 76, page 195 in [4], we have in the quasi-static limit:

$$\mathrm{E}_y = -\int_0^\infty J_0(\lambda y) e^{-\lambda h} \lambda d\lambda = -\frac{h}{(h^2 + y^2)^{3/2}}. \tag{56}$$

From [5], pages 98-101, we see that the result in the quasi-static limit should be:

$$\mathrm{E}_y = -\frac{j\omega\mu_0}{4\pi r^2} \mathrm{I}A \sin\theta \cos\phi = -\frac{j\omega\mu_0 h}{4\pi (y^2 + h^2)^{3/2}} \mathrm{I}A, \tag{57}$$

where **r** is a vector from the magnetic dipole to the point in question, θ is the angle this makes with the dipole axis (90º) and ϕ is the angle between the electric field vector and the y-axis. Thus, the constant of proportionality is:

$$\frac{j\omega\mu_0 \mathrm{I}A}{4\pi}. \tag{58}$$

The full expression for the electric field at the wire can now be written:

$$\mathrm{E}_y(y) = \frac{j\omega\mu_0 \mathrm{I}A}{4\pi} \int_0^\infty J_0(\lambda y)\lambda d\lambda \left(e^{u(a-h)} - \frac{n^2 u - u_E}{n^2 u + u_E} e^{-u(a+h)} \right). \tag{59}$$

This can be evaluated numerically without serious difficulty.

Free-Space Comparison

A comparison between FEKO and the analytic approach is useful to verify the results from both methods. FEKO is a program that uses purely numerical techniques based on Maxwell's equations to simulate the current in a wire. The FEKO results are described in [1]. The basic parameters for free-space simulations involving bare and insulated wires are provided in Table 2. As noted, the magnetic dipole lies above the wire centre and its axis is perpendicular to the wire.

The electric field along the wire is plotted in Figure 8; this has been derived by numerical integration using (59). The real part of the complex field is almost zero and the imaginary part is only significant close to the magnetic dipole. In this case, a full-wave formula derived from the magnetic vector potential arising from a magnetic dipole in a uniform isotropic medium [5] is a useful close approximation; this is:

$$E = -\frac{j\omega\mu_0 IA}{4\pi} jk \left(1 + \frac{1}{jkr} \right) \frac{e^{-jkr}}{r} \sin\theta. \tag{60}$$

Here E is the field parallel to the wire, k is the angular wave number in the medium, r is the position vector from the dipole to the point in question on the wire and θ is the angle this vector makes with the dipole axis. The numerical and analytic results from (59) and (60) appear to be practically the same.

Table 2
Free-Space Basic Parameters

Parameter	Value
Frequency (MHz)	0.5, 5.0, 50.0
Dipole Strength (Am^2)	0.0029
Dipole Height (m)	0.10
Bare Wire Height (mm)	0.50*
Wire Length (m)	60.0
Wire Radius (mm)	0.50
Insulation Relative Permittivity	4.0
Insulation Radius (mm)	1.5

* Increased to 1.5 mm when insulated

The quasi-static field based on (57) is virtually indistinguishable from the previous results. This is to be expected because the wavelength of 60 m is much greater than the distances involved in the plot (< 1 m). Therefore it can be concluded that the quasi-static approximation is appropriate in this situation. Moreover, because the electric field is significant only close to the magnetic dipole, the quasi-static approximation should often be applicable to a non-magnetic half space, which represents a great simplification.

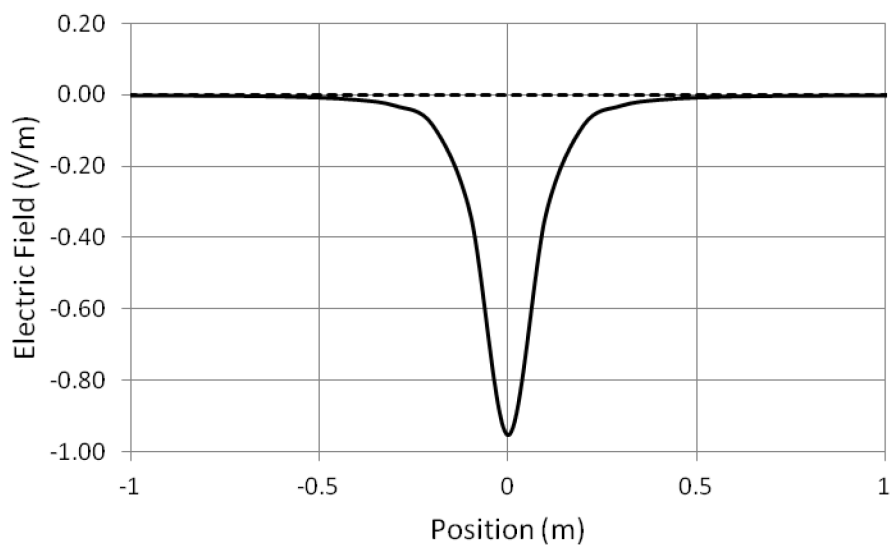

Figure 8. The free-space complex electric field along the wire (solid imaginary, dashed real).

Though the propagation constant for the wire in free-space is the same as the propagation constant for electromagnetic waves in free-space, the transmission line model requires the

characteristic impedance. This can be estimated from the capacitance of the wire, for example by regarding it as a prolate spheroid. The capacitance per unit length for a thin wire of length l and radius a is given approximately by [6] (page 74):

$$C = \frac{2\pi\varepsilon_0}{\log(l/a)} \tag{61}$$

The inductance per unit length is found using (16) and this involves a similar logarithmic term.

When the wire is insulated, the inductance associated with the insulation is added to that of the medium. Similarly, the capacitance per unit length is calculated as the result of the capacitance across the insulation in series with the capacitance of a wire with the radius of the insulation; the results are inserted into (16) to find the propagation constants.

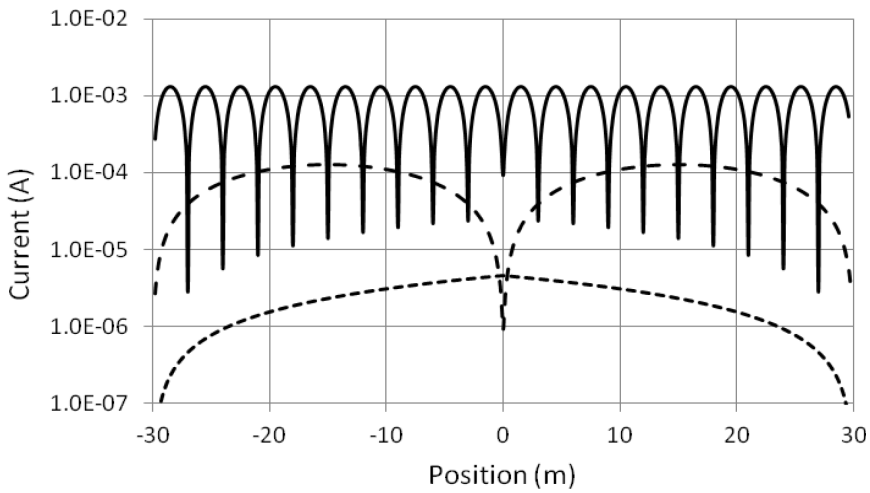

Figure 9. Bare wire response at 0.5 MHz (small dash), 5 MHz (long dash) and 50 MHz (solid).

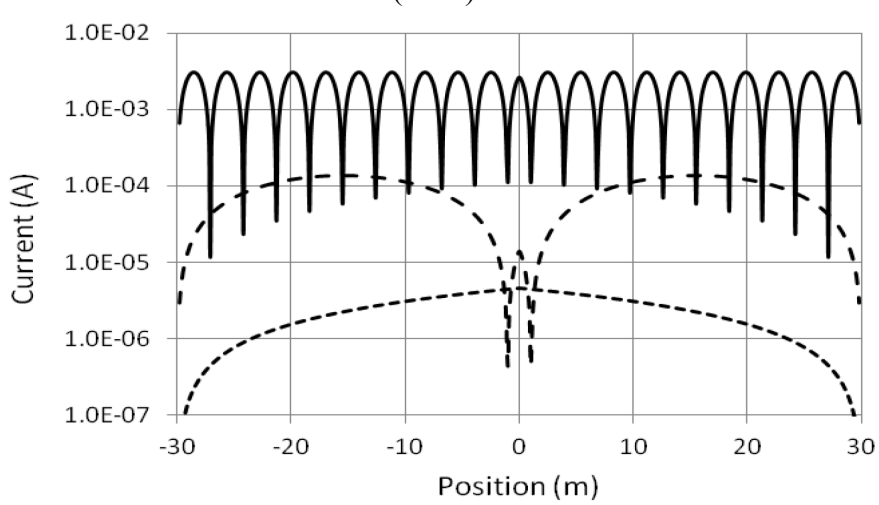

Figure 10. Insulated wire response at 0.5 MHz (small dash), 5 MHz (long dash) and 50 MHz (solid).

The impressed electric field induces a current in the wire. This is treated using a Green's function approach (see Appendix C).

Figures 9 and 10 show the bare and insulated wire responses at three frequencies (0.5, 5.0, 50.0 MHz). The parameters are in Table 2; the insulation radius for Figure 10 is 1.5 mm and its relative permittivity is 4.0.

With the exception of the 50 MHz response of an insulated wire, the responses are typically within 30% of the FEKO responses. Problems arise at frequencies that result in many wavelengths along the wire. Then end effects are likely to play an important role; a slight difference between the effective length of the wire and its actual length become very important, especially close to resonance frequencies as in the 5 MHz and 50 MHz cases. It is well-known that the resonant length of a thin half wave dipole is a few percent less than its actual length. Thus, a small change in the length of the wire used in the simulation can correct the discrepancy.

Full-Space Comparison

Figures 11 and 12 show the responses for bare and insulated wires using the parameters in Table 2 for a frequency of 5 MHz, a relative permittivity of 2.5 and conductivities ranging from 0 S/m to 10^{-2} S/m. These are similar to the FEKO results in both shape and magnitude. When the conductivity of the medium is large the bare wire response tends to a straight line on either side of the excitation point. This is characteristic of a damped response with almost no signal reaching the wire end so that there are negligible reflections. As noted previously, the conducting dielectric does not change the quasi-static electric field near the wire. This is because the primary field from a magnetic dipole, which of course is magnetic, produces an electric field by virtue of Faraday's law. The electric fields produce eddy currents that affect the resulting electromagnetic field but these are much smaller than the primary fields and can be neglected in the present context.

As with the wire in free space, the capacitance per unit length is affected by the insulation. When the medium is conducting, the insulation also affects the inductance per unit length and this must be recalculated by considering the total inductance as the sum of the inductance due to the insulation in series with the inductance due to the medium. If required, the internal inductance of the wire can also be added.

As with the non-magnetic medium, the presence of insulation around the wire does not affect the impressed electric field. However, insulation alters the induced line current characteristics by changing the electrical properties of the wire, which we regard as a transmission line. When the medium is conducting, low permittivity insulation tends to raise the characteristic impedance of the line and reduce the induced current relative to the bare wire. Increasing the conductivity tends to reduce the characteristic impedance, which raises the induced current. These changes also affect the propagation constant of the line and these can result in shifts into or out of resonance.

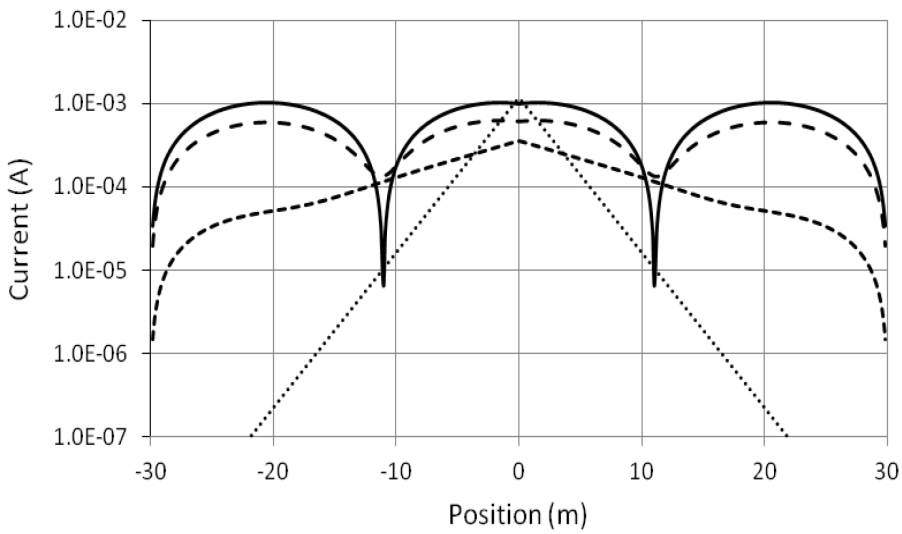

Figure 11. Full space bare wire response at 5 MHz for conductivities of 0 S/m (solid line), 10^{-4} S/m (large dash), 10^{-3} S/m (small dash) and 10^{-2} S/m (dotted).

The effects can be observed in Figure 12. For example, when the conductivity of the medium is high, the insulation around the wire reduces the effect of the conductivity and the attenuation of the signal as it propagates away from the excitation. When the conductivity is zero, the insulation changes the propagation constant so that the wire is resonant. The node depth is higher than in the FEKO simulations. Otherwise the results are very close.

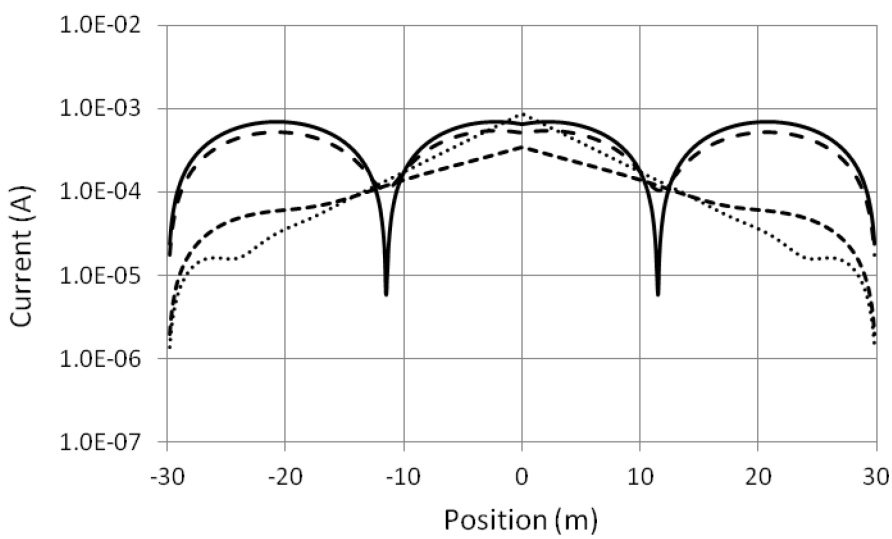

Figure 12. Full space insulated wire response at 5 MHz for conductivities of 0 S/m (solid line), 10^{-4} S/m (large dash), 10^{-3} S/m (small dash) and 10^{-2} S/m (dotted).

Half-Space Comparison

Using the details in Appendix B, the expressions for the inductance and capacitance per unit length of the transmission line can be simplified and are given by the following formulae:

$$L = \begin{cases} \dfrac{\mu_0}{2\pi}\left(\log\left(\dfrac{2h+a}{a}\right) + 2\displaystyle\int_0^\infty \dfrac{e^{-u(2h+a)}}{u+u_E} d\lambda \right), & h > 2a \\ \dfrac{\mu_0}{2\pi}\left(-\dfrac{j\pi}{2}\left(H_0^{(2)}(k_E a) - H_0^{(2)}(k_E(2h+a))\right) + 2\displaystyle\int_0^\infty \dfrac{e^{-u_E(2h+a)}}{u+u_E} d\lambda \right), & h < -2a \end{cases}$$

$$C = \begin{cases} 2\pi\varepsilon_0\left(\log\left(\dfrac{2h+a}{a}\right) + 2\displaystyle\int_0^\infty \dfrac{e^{-u(2h+a)}}{u+\varepsilon_E u_E} d\lambda \right)^{-1}, & h > 2a \\ 2\pi\varepsilon_E\varepsilon_0\left(-\dfrac{j\pi}{2}\left(H_0^{(2)}(k_E a) - H_0^{(2)}(k_E(2h+a))\right) + 2\displaystyle\int_0^\infty \dfrac{\varepsilon_E e^{-u_E(2h+a)}}{u+\varepsilon_E u_E} d\lambda \right)^{-1}, & h < -2a \end{cases} \tag{62}$$

In the domain where $-2a < h < 2a$, the wire and its image are close together and these expressions are not even approximately valid. Therefore interpolation must be employed to cover this interval.

As noted previously, the integrands of the integrals tend to peak where u and u_E go to zero. To avoid problems with a rapidly varying integrand, the path of integration can be moved away from the real axis as the poles are approached. In this study, the path starts at the origin and then moves linearly at an angle into the upper half of the complex plane. After it passes the last pole, it moves parallel to the real axis out to infinity.

In the half-space simulation the wire lies on the ground, which has a relative permittivity of 2.5. The impressed electric field is the quasi-static version because, as in the previous cases it is highly localized near the wire centre. The current in the wire is calculated using the transmission line model, which requires the propagation constant and the characteristic impedance. In a previous section, using a frequency of 5 MHz and a conductivity of 10^{-3} S/m, it is found that the propagation constant is approximately (0.15 - j0.5) rad/m and the characteristic impedance is approximately (370 + j90) Ohm. The current in the wire is shown in Figure 13. For other conductivities, the propagation constants are shown in Table 3.

Table 3.
Half-Space Propagation Parameters

Conductivity (S/m)	Wavenumber (rad/m)	Characteristic Impedance (Ohm)
1.0E-6	0.135 - 0.0j	475-75j
1.0E-4	0.135 - 0.0075j	470-50j
1.0E-3	0.150 - 0.05j	380+90j
1.0E-2	0.260 – 0.11j	200 + 35j

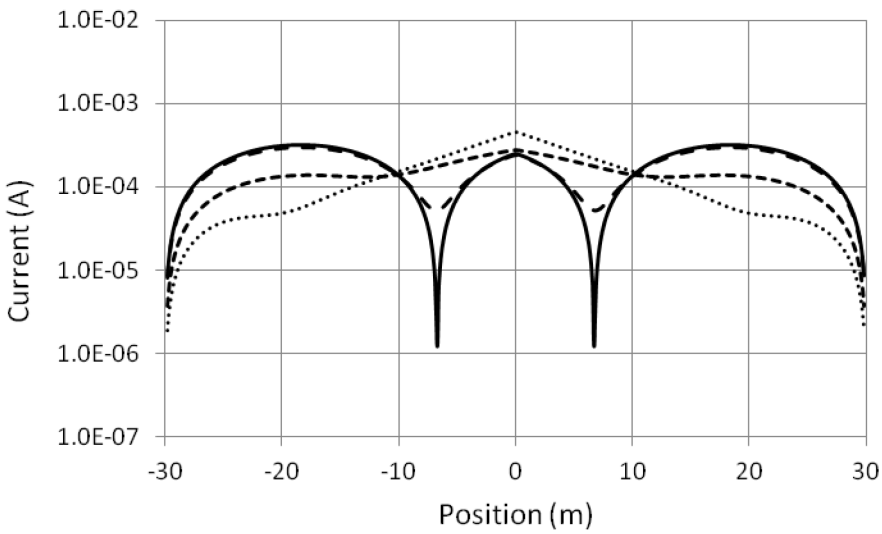

Figure 13. Half space bare wire response at 5 MHz for conductivities of 10^{-6} S/m (solid line), 10^{-4} S/m (large dash), 10^{-3} S/m (small dash) and 10^{-2} S/m (dotted).

These graphs are very close to the FEKO curves except near the nodes, which seem to be exaggerated.

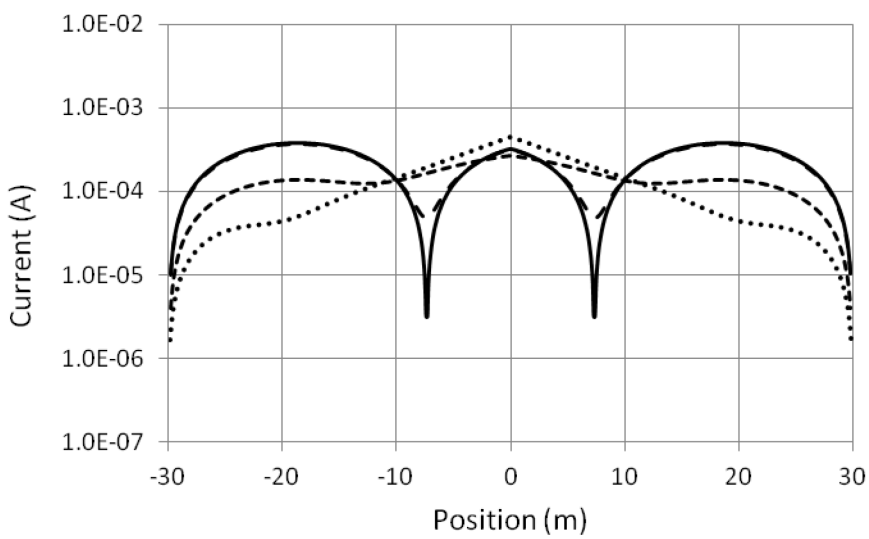

Figure 14. Half space insulated wire response at 5 MHz for conductivities of 10^{-6} S/m (solid line), 10^{-4} S/m (large dash), 10^{-3} S/m (small dash) and 10^{-2} S/m (dotted).

Conclusions

The transmission line model can provide useful engineering results for the current induced in a thin wire located near a conducting half space. When the medium and the ground are non-magnetic and the excitation is from a magnetic dipole, a simple formula for the electric field can be employed. In cases where the excitation is from a magnetic

dipole located close to the wire compared to a wavelength, an acceptable impressed electric field can be derived by using quasi-static approximation. In this case, the field is not affected significantly by insulation on the wire or by the complex permittivity of the medium. When the wire is near a half space and the quasi-static approximation is not appropriate, it is necessary to estimate the impressed electric field using the Sommerfeld theory. It is worth noting that the situation is more complicated for electric dipole excitation.

For the ground parameters used here, the induced currents using the transmission line approach and those using the FEKO numerical simulation are typically similar to within some tens of percent but there can be some significant discrepancies partly due to the inability of the transmission line approach to account properly for end effects. The transmission line model does not handle the resonant condition well, especially at high frequencies. Nevertheless, it may be possible to correct this problem using an empirical generalization of the resonant length of a half-wave dipole.

Though the model appears to provide satisfactory results for the parameter set chosen, there are indications that the model may start to fail when the ground complex permittivity is high. More study is needed to explore the limits of validity and how the parameter range might be extended.

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Appendix A: Electric Field in a Half Space

We illustrate the theory for the case of a point charge placed in air (or vacuum) above a simple non-conducting dielectric; the interface is horizontal. It is well-known that at low frequencies the effect of the dielectric can be represented by image charges placed under and above the interface. The relative permittivity is real and there are no significant magnetic fields so that the theory does not require a Hertz vector (or a magnetic potential vector) and can be framed in terms of scalar potentials.

The problem exhibits an azimuthal symmetry about the vertical axis and we adopt a cylindrical coordinate system where the z-axis is vertical. The fields above the interface can be expressed as the sum of a primary potential V_P, which is equal to the potential that would exist in the absence of the dielectric, and a secondary potential, V_S, which represents the influence of the dielectric. Below the interface, the potential is V_E. Apart from a constant factor and employing the same concepts as Sommerfeld, the primary potential for a charge at height, h, is given for $z < h$ by:

$$V_P = \frac{e^{-jkR}}{R} = \int_0^\infty J_0(\lambda r) e^{u(z-h)} \frac{\lambda d\lambda}{u} \tag{63}$$

The secondary potential is given in terms of an unknown function $F(\lambda)$:

$$V_S = \int_0^\infty F(\lambda) J_0(\lambda r) e^{-u(z+h)|} \frac{\lambda d\lambda}{u} \tag{64}$$

Similarly, the potential beneath the surface is:

$$V_E = \int_0^\infty F_E(\lambda) J_0(\lambda r) e^{u_E z - uh} \frac{\lambda d\lambda}{u} \tag{65}$$

Note that, for convenience, we have introduced an additional factor u^{-1} in the expressions for V_S and V_E.

There are two boundary conditions that can be employed to eliminate the unknown functions F and F_E. The transverse components of the electric field across the interface must be equal and the vertical components of the electric displacement on each side of the interface must be the same. The electric field is just given by:

$$\mathbf{E} = -\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \tag{66}$$

Therefore the radial component at $z = 0$ just above the dielectric is:

$$\mathrm{E}_r = \int_0^\infty J_1(\lambda r) e^{-uh} \frac{\lambda^2 d\lambda}{u}(1 + F(\lambda)) \tag{67}$$

Just below the interface we have:

$$\mathrm{E}_r = \int_0^\infty J_1(\lambda r) e^{-uh} \frac{\lambda^2 d\lambda}{u} F_E(\lambda) \tag{68}$$

Therefore F and F_E are related by:

$$1 + F = F_E \tag{69}$$

The normal component of the displacement can be treated in the same way by considering the z-component. This provides another relation:

$$\varepsilon_1 \varepsilon_0 u(1-F) = \varepsilon_2 \varepsilon_0 u_E F_E \tag{70}$$

Where $\varepsilon_{1,2}$ are the relative permittivities of the upper and lower half spaces. The solution is:

$$\begin{aligned} F &= \frac{\varepsilon_1 u - \varepsilon_2 u_E}{\varepsilon_1 u + \varepsilon_2 u_E} \\ F_E &= \frac{2\varepsilon_1 u}{\varepsilon_1 u + \varepsilon_2 u_E} \end{aligned} \tag{71}$$

The function F represents the potential for a charge located at $z = -h$ rather than $z = h$. This is an image of the original charge. F is complicated except in the low frequency limit when u, $u_E \rightarrow \lambda$. Then, if the upper half space is air, F and F_E become:

$$\begin{aligned} F &= \frac{1-\varepsilon_2}{1+\varepsilon_2} \\ F_E &= \frac{2\varepsilon_2}{1+\varepsilon_2} \end{aligned} \tag{72}$$

Thus the electrostatic field in air is found to be that from the original charge, q, in the absence of a dielectric, plus that from an image charge qF at an equal distance below the dielectric surface. The field within the dielectric is attributable to a charge qF_E located at the same position as the charge q. These final results are identical to those described by Bleaney and Bleaney ([7] page 55).

The result can be generalized to a line charge in the same way as for the current element.

Appendix B: Hankel Functions

The Hankel functions are defined in terms of Bessel functions of the first and second kind [4]:

$$H_n^{(1)}(z) = J_n(z) + jY_n(z)$$
$$H_n^{(2)}(z) = J_n(z) - jY_n(z) \tag{73}$$

When the time dependence is of the form $e^{j\omega t}$, outward traveling waves are represented by $H^{(2)}$ but, as is clear from the definition, Sommerfeld's outward waves are represented by $H^{(1)}$.

McLachlan ([4] page 199) shows that a Hankel function can be expressed in terms of an integral of the type considered here, i.e.

$$H_0^{(1),(2)}(a\sqrt{z^2+b^2}) = \mp\frac{1}{j\pi}\int_{-\infty}^{\infty}\frac{e^{\pm b\lambda - z\sqrt{\lambda^2-a^2}}}{\sqrt{\lambda^2-a^2}}d\lambda, \quad a,b,z \text{ real} > 0 \tag{74}$$

Therefore we expect that, when the range of integration is over [0, ∞], we should have:

$$H_0^{(1),(2)}(a\sqrt{z^2+b^2}) = \mp\frac{2}{j\pi}\int_{0}^{\infty}\frac{e^{\pm b\lambda - z\sqrt{\lambda^2-a^2}}}{\sqrt{\lambda^2-a^2}}d\lambda, \quad a,b,z \text{ real} > 0 \tag{75}$$

It is important to note that the parameters are all real. However, it should be possible to generalize this result for complex parameters using arguments based on analytic continuity. The problem then becomes one of choosing the appropriate square roots by locating the appropriate Riemannian sheet. The latter appears to be defined by the range of phase of [-π, π]. For example:

$$\int_0^{\infty}\frac{e^{-z\sqrt{\lambda^2-k^2}}}{\sqrt{\lambda^2-k^2}}d\lambda = -\frac{j\pi}{2}H_0^{(2)}(kz), \quad z\ real > 0 \tag{76}$$

We can differentiate with respect to z and this yields Hankel functions of order 1:

$$\int_0^{\infty} e^{-z\sqrt{\lambda^2-k^2}}d\lambda = -\frac{j\pi k}{2}H_1^{(2)}(kz), \quad z\ real > 0\,. \tag{77}$$

These results have been verified numerically. The sign is determined by the sign of the imaginary part of k; if this is negative, the Hankel function is of the second type.

Fig. 15 shows typical plots of the integrals resulting in Hankel functions calculated directly using series expansions coupled with asymptotic expressions for large argument. The integrals can be computed very quickly for typical soil wave numbers and are typically accurate to about 1%; their plots are indistinguishable from those derived directly.

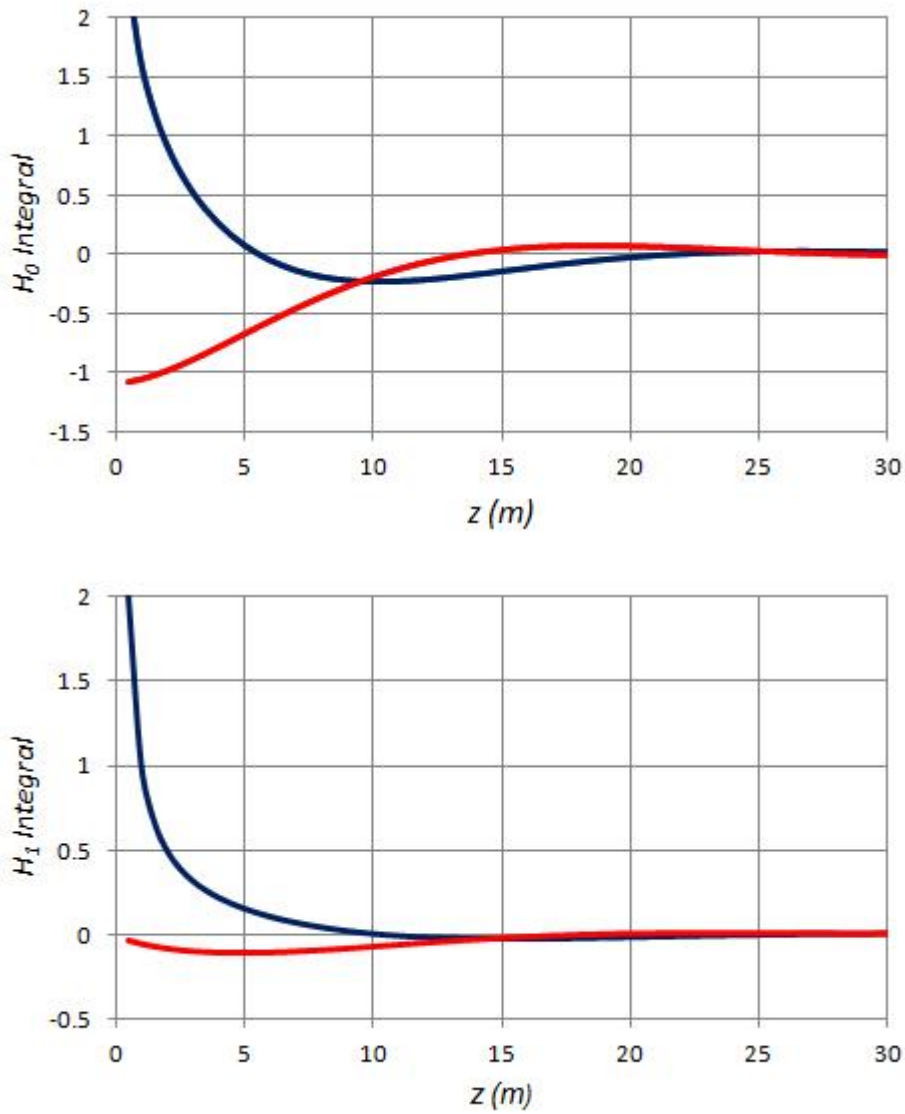

Figure 15. Plots of integrals in (64) and (65) (real part blue; imaginary part red). The relative permittivity is 2.5, the conductance is 10^{-3} S/m and the frequency is 5 MHz; these give k = (0.194 – j0.102) rad/m.

In the present context the wire radius is very small compared with the propagation wavelength so that the argument of the Hankel function will often be small. Therefore it is useful to examine how the Hankel function behaves for small arguments. By reference to McLachlan [4] we see that in the limit as z approaches zero:

$$J_0(z) \to 1.0, \quad J_1(z) \to z/2,$$
$$Y_0(z) \to \frac{2}{\pi}\log\frac{z}{2}, \quad Y_1 \to -\frac{2}{\pi z} \tag{78}$$

We conclude that in the same limit the Hankel function behavior is:

$$H_0^{(2)}(z) \to -\frac{2j}{\pi}\log\frac{z}{2}, \quad H_1^{(2)}(z) \to \frac{2j}{\pi z} \tag{79}$$

Then we find that:

$$\int_0^\infty \frac{e^{-z\sqrt{\lambda^2-k^2}}}{\sqrt{\lambda^2-k^2}}\,d\lambda = -\frac{j\pi}{2}H_0^{(2)}(kz) \to -\log\frac{kz}{2},$$
$$\int_0^\infty e^{-z\sqrt{\lambda^2-k^2}}\,d\lambda = -\frac{j\pi k}{2}H_1^{(2)}(kz) \to \frac{1}{z} \tag{80}$$

Appendix C: Current Green's Function

The electric field at a wire induces a current that depends on how the electric field varies along the wire and how the wire, regarded as a transmission line, reacts to the field. This can be handled using a Green's function approach in which the current distribution is an integral over the field along the wire. This method is used by Hill [8] and a simple derivation is provided here.

The current, i, and voltage, v, along an infinite transmission line are related by:

$$v = i / Z_0 \tag{81}$$

where Z_0 is the characteristic impedance of the line. This can be found by considering the input impedance of an infinite line and then adding another section to it, which leaves the impedance unaltered. When a generator applies an oscillating electromotive force, e, to the end of the line, a wave propagates down the line with propagation constant, k. The voltage at a point on the line is given by:

$$v = v_0 e^{j(\omega t - kx)} \tag{82}$$

where ω is the angular frequency of the generator.

Next we consider a line terminated by an open circuit as in Figure 16.

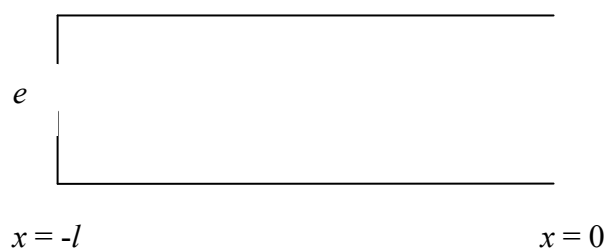

Fig. 16. Open circuit line.

The current at $x = 0$ must be zero. This condition can be satisfied by introducing a wave of equal amplitude but traveling in the opposite direction, i.e.

$$i = A(e^{j(\omega t - kx)} - e^{j(\omega t + kx)}) \tag{83}$$

where A is a constant. The voltage is now the sum of the voltages of each wave:

$$v = Z_0 A(e^{j(\omega t - kx)} + e^{j(\omega t + kx)}) \tag{84}$$

When the generator is located at $x = -l$ and omitting the frequency factor, we have:

$$e = Z_0 A(e^{jkl} + e^{-jkl}) \tag{85}$$

This determines the constant A, so that the current on the line is:

$$i = \frac{e}{Z_0} \frac{e^{-jkx} - e^{jkx}}{e^{jkl} + e^{-jkl}} = \frac{je}{Z_0} \frac{\sin(kx)}{\cos(kl)} \tag{86}$$

The problem that we need to solve is the case where an infinitesimal source of *emf* is inserted into one wire of a transmission line as in Figure 17. This shows the nominal directions of the currents and *emf*s; the line is of length 2*h* and the *emf* is located at *x*.

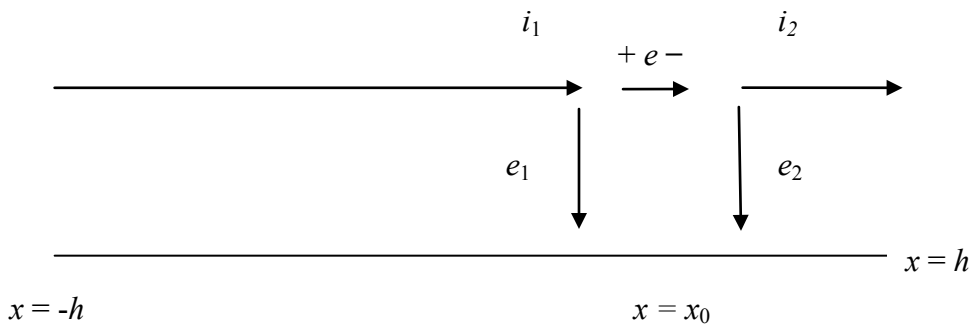

Fig. 17. Open circuit line with infinitesimal impressed emf.

At the location of the infinitesimal generator we have:

$$\begin{aligned} i_1 &= i_2 = i_0 \quad \text{(say)} \\ e &= e_1 - e_2 \\ i_1 &= -\frac{je_1}{Z_0}\frac{\sin(k(h+x_0))}{\cos(k(h+x_0))} \\ i_2 &= \frac{je_2}{Z_0}\frac{\sin(k(h-x_0))}{\cos(k(h-x_0))} \end{aligned} \tag{87}$$

These equations can be solved by eliminating i_1, i_2, e_1 and e_2 to yield the current at the source of *emf*:

$$i_0 = \frac{je}{Z_0}\frac{\sin(k(h+x_0))\sin(k(h-x_0))}{\sin(2kh)} \tag{88}$$

Because of the factor $\sin(k(h - x_0))$, which plays the same role as $\sin(kx)$ in (85), this applies not only to the current at x_0 but also to currents to the right of the generator.

It follows that the Green's function for the current distribution is given by:

$$G(x,x') = \begin{cases} \dfrac{j}{Z_0}\dfrac{\sin(k(h+x'))\sin(k(h-x)}{\sin(2kh)} & x > x' \\ \dfrac{j}{Z_0}\dfrac{\sin(k(h+x))\sin(k(h-x')}{\sin(2kh)} & x < x' \end{cases} \tag{89}$$

and

$$i(x) = \int_{-h}^{h} e(x')G(x,x')dx' \tag{90}$$

Appendix D: Wire Internal Inductance

Inside the wire, the component of the current density, **J**, parallel to the wire along the *z*-axis satisfies Bessel's equation [3]. When the wire conductivity is σ_c and neglecting the relative permittivity of the wire, this becomes:

$$\frac{\partial^2 J_z}{\partial r^2} + \frac{1}{r}\frac{\partial J_z}{\partial r} - j\sigma_c \mu_0 \omega J_z = 0 \quad . \tag{91}$$

The solution is:

$$J_z = AJ_0(k_c r) \tag{92}$$

where A is a constant of integration, J_0 is a Bessel function of the first kind and k_c is a wave number in the wire and is given by:

$$k_c = (-j\sigma_c \mu_0 \omega)^{1/2} . \tag{93}$$

The constant of integration can be found by expressing the current density in terms of the total current in the wire, I_s. By integrating, we have:

$$I_s = A\int_0^a 2\pi r J_0(k_c r)dr = 2\pi a A J_1(k_c a)/k_c \tag{94}$$

Therefore we have:

$$J_z(r) = \frac{k_c I_s}{2\pi a}\frac{J_0(k_c r)}{J_1(k_c a)} \tag{95}$$

The internal impedance per unit length, Z_{int}, of the wire is given by:

$$Z_{\text{int}} = \frac{E_s}{I_s} = \frac{J_z(a)/\sigma_c}{I_s} = \frac{k_c J_0(k_c a)}{2\pi a \sigma_c J_1(k_c a)}, \tag{96}$$

where E_s is the electric field component parallel to the *z*-axis at the surface of the wire.

Thus the complex internal inductance is given by:

$$L = \frac{Z_{\text{int}}}{j\omega} = \frac{1}{2\pi a}\left(\frac{j\mu_0}{\omega\sigma_c}\right)^{1/2}\frac{J_0(k_c a)}{J_1(k_c a)} \tag{97}$$

Equations (95) and (96) can be cast into different forms. For example, using relationships in [4], an equivalent to (95) is:

$$Z_{\text{int}} = \frac{1}{2\pi a}\left(\frac{j\omega\mu_0}{\sigma_c}\right)^{1/2}\frac{I_0(jk_c a)}{I_1(jk_c a)} \tag{98}$$

Where I_0 and I_1 are modified Bessel functions of the first kind.

The square roots in the expressions for the internal impedance and the inductance contain a square root and this introduces a sign ambiguity. It is important to check that the result of calculations corresponds to a physically realizable result. The real part of the impedance and inductance should be positive; the imaginary part of the impedance should be positive corresponding to inductance and the imaginary part of the inductance should be negative again corresponding to resistance.