A Summary of EM Theory for Dipole Fields near a Conducting Half-Space

James K.E. Tunaley

Abstract—**Estimates of the electric and magnetic fields arising from an impressed current source above or within the ground are important for estimating the detectability of wires. Important contributions to the theory have been made by Sommerfeld, Banos, Wait and King. However, some of the basic concepts are described in books that are over 50 years old and are becoming difficult to access. Therefore the concepts, as described by the first three authors, are summarized.**

Index Terms—**Electomagnetic Propagation, Conducting Half-Space, Buried Wire Detection.**

I. Introduction

THE application of electromagnetic theory to estimate the fields from conducting wires close to or beneath the ground is important for the detection of Improvised Explosive Devices (IEDs), which are often triggered by command wires. The command wires can be excited by external radio frequency signals and detected by small changes in the scattered fields close to the wire.

The theoretical basis is usually that adopted by Sommerfeld in his pioneering work on the propagation of radio broadcasts over the conducting earth. The finite conductivity of the earth is important because the propagating waves induce currents in the earth. These create a component of the electric field parallel to the ground and permit the transmission of signals over much greater distances than if the earth were infinitely conducting.

Mainly two groups of researchers have been involved with the theory of fields from horizontal wires near a conducting interface. These groups were headed by James Wait and Ronold King. There were also several important independent workers, such as Carson and Banos: here I shall focus primarily on the theoretical basis using the results of Sommerfeld, Banos and Wait. Their papers often refer back to early work and the details of basic concepts are not repeated in recent literature.

Unfortunately much of the early theory is located in books, which are becoming difficult to obtain by the independent researcher, and that is the reason for this summary. I also add explanatory comments where I feel it is appropriate.

J.K.E. Tunaley is President of London Research and Development Corporation, Ottawa, ON, and Prof. Emeritus, Physics Dept., University of Western Ontario, London.
Email: manager@london-research-and-development.com
Published January 20th, 2012.

The basic theory is actually fairly straightforward but is algebraically quite tedious especially when matching the boundary conditions at an interface.

II. Sommerfeld Theory

Sommerfeld deals with radio propagation over the ground in chapter 6 of his book on partial differential equations [1]. His treatment applies to the Hertzian dipole (comprising two very short elements of constant current) and is based on the Hertz potential, $\mathbf{\Pi}$. In the present context we consider an oscillatory excitation so that, apart from a constant of proportionality, this potential can be identified as the magnetic vector potential, **A**. The solutions of the electrodynamics equations for an oscillating charge in free space are assumed to be of the form:

$$\Pi = \frac{1}{r} e^{j(kr-\omega t)} \tag{1}$$

and the fields can be obtained by differentiation. The actual relation between the two potentials is:

$$\mathbf{\Pi} = \frac{j\omega}{k^2} \mathbf{A} \tag{2}$$

In the presence of a conducting ground, the general concept is to treat the Hertz potential as the sum of two parts. The first part represents the field created by the elementary current in the dipole in free space without the presence of the ground. The second can be regarded as the contribution due to the ground and includes the effects of induced currents. The form of the second part is derived by matching the tangential electric and magnetic fields at all points across the plane boundary. These are initially unknown and the key to the solution is to express the two Hertz potentials as Fourier-Bessel transforms both of which are in the form of an integral. It then proves to be possible to apply the boundary conditions in a straightforward manner.

Sommerfeld shows that:

$$\begin{aligned} \mathbf{E} &= k^2 \mathbf{\Pi} + \nabla\nabla \cdot \mathbf{\Pi} \\ \mathbf{H} &= \frac{k^2}{\mu\mu_0 j\omega} \nabla \times \mathbf{\Pi} \end{aligned} \tag{3}$$

These relations are similar to those involving the magnetic vector potential in which the Lorentz gauge condition is invoked to specify **A** in terms of an electric potential, Φ, i.e.

$$\nabla \cdot \mathbf{A} = \frac{j\omega}{c^2} \Phi \tag{4}$$

As with **A**, both the electric and magnetic fields can be derived from just the Hertz potential and the electric potential Φ is not required.

In general, the angular wave number, k, is given by:

$$k^2 = \varepsilon\mu\omega^2 + j\mu\sigma\omega \quad (5)$$

where ε and μ are the actual permittivity and permeability of the medium and σ is the conductivity.

The Hertz potential satisfies the equation:

$$\nabla^2\mathbf{\Pi} = \left(\varepsilon\mu\frac{\partial^2}{\partial t^2} + \sigma\mu\frac{\partial}{\partial t}\right)\mathbf{\Pi} \to -k^2\mathbf{\Pi} \quad (6)$$

When this wave equation is written out in cylindrical coordinates and $\partial/\partial\phi = 0$ (such as for the vertical dipole), we have for the z-component of the Hertz potential:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2 + \frac{\partial^2}{\partial z^2}\right)\Pi_z = 0 \quad (7)$$

The eigenfunctions, u, can be determined by a separation of variables and so we have:

$$u = J_0(\lambda r)\cos(\mu z), \quad k^2 = \lambda^2 + \mu^2 \quad (8)$$

where μ is no longer the permeability but is given by:

$$\mu = \frac{m\pi}{h} \quad (9)$$

and h is the length of the cylinder on which the normal derivative is zero; m is an integer. The length of the cylinder is now allowed to become infinite so that we have a continuous distribution of eigenfunctions and:

$$\Pi_z = \int_0^\infty F(\lambda)J_0(\lambda r)e^{\pm\mu z}d\lambda, \quad \mu = \sqrt{\lambda^2 - k^2} \quad (10)$$

where the cosine is replaced by exponentials. The function F has to be determined.

For a source located at $z = 0$, the Hertz potential is given by:

$$\Pi_z = \frac{e^{jkR}}{R}, \quad R^2 = r^2 + z^2 \quad (11)$$

By performing an inverse transformation, Sommerfeld then shows that, for this case, the function, F, is given by:

$$F(\lambda) = \frac{\lambda}{\mu} \quad (12)$$

and therefore:

$$\Pi_z = \int_0^\infty J_0(\lambda r)e^{-\mu|z|}\frac{\lambda d\lambda}{\mu} = \frac{1}{2}\int_{-\infty}^\infty H_0^{(1)}(\lambda r)e^{-\mu|z|}\frac{\lambda d\lambda}{\mu} \quad (13)$$

The parameter, μ, which plays the role of a propagation constant in the z-direction, is ambiguous because of the square root; the sign with a positive real part must be taken.

When an antenna is located above an infinitely conducting earth, the effect of boundary conditions at the earth surface can be represented by images. The image of a vertical dipole is another vertical dipole with its polarity in the same direction as the actual dipole. Therefore, if the actual dipole is located very close to the surface, the Hertz potential is doubled. In contrast, the polarity of the image of a horizontal dipole is opposite to that of the actual dipole. When a horizontal dipole is placed close to the surface, cancellation occurs; the system and its fields can be represented by a quadrupole if needed.

When a vertical dipole is located above a conducting earth, we augment the permittivity in the ground by a complex conductivity. Thus we have two media and in the conducting medium, currents are induced that give rise to secondary fields. In air, the primary potential is given by (13) and there is a secondary potential from the ground:

$$\Pi_z = \Pi_{prim} + \Pi_{\sec} = \begin{cases} \int_0^\infty J_0(\lambda r)d\lambda\left(e^{-\mu(z-h)}\frac{\lambda}{\mu} + e^{-\mu(z+h)}F(\lambda)\right), & z > h \\ \int_0^\infty J_0(\lambda r)d\lambda\left(e^{\mu(z-h)}\frac{\lambda}{\mu} + e^{-\mu(z+h)}F(\lambda)\right), & h > z > 0 \end{cases} \quad (14)$$

In the earth, there is no primary potential and we have:

$$\Pi_z = \Pi_E = \int_0^\infty J_0(\lambda r)d\lambda\, e^{\mu_E z - \mu h}F_E(\lambda), \quad \mu_E = \sqrt{\lambda^2 - k_E^2}, \quad 0 > z > -\infty \quad (15)$$

Notice that the terms F and F_E have been multiplied by $e^{-\mu h}$. This is simply for convenience (μ is a function of λ so that the factor can be absorbed into F's).

To determine F and F_E, the tangential components of **E** and **H** must be continuous across the air-ground interface. For the vertical dipole, Sommerfeld shows that the continuity conditions at $z = 0$ are:

$$\frac{\partial\Pi}{\partial z} = \frac{\partial\Pi_E}{\partial z}, \quad \Pi = \frac{k_E^2}{k^2}\Pi_E \quad (16)$$

These lead immediately to:

$$\mu F + \mu_E F_E = \lambda, \quad \mu F - \frac{k_E^2}{k^2}\mu F_E = -\lambda \quad (17)$$

Solving the two equations yields:

$$F = \frac{\lambda}{\mu}\left(1 - \frac{2\mu_E}{\mu k_E^2/k^2 + \mu_E}\right), \quad F_E = \frac{2\lambda}{\mu k_E^2/k^2 + \mu_E} \quad (18)$$

Sommerfeld demonstrates that, when the ground becomes highly conducting, the Hertz potential reduces to that predicted by image theory.

For the horizontal dipole, the calculation is more complicated because, according to Sommerfeld, two components of the Hertz potential must be considered. The dipole lies in the x-direction and the two components are Π_x and Π_z. This choice can be justified by the fact that it is possible to satisfy the boundary conditions and because of the uniqueness of the solutions to the wave equation.

The boundary conditions for the electric field become:

$$\nabla\cdot\mathbf{\Pi} = \nabla\cdot\mathbf{\Pi}_E, \quad k^2\Pi_x = k_E^2\Pi_{Ex} \quad (19)$$

The boundary conditions for the magnetic field become:

$$\Pi_z = \frac{k_E^2}{k^2}\Pi_{Ez}, \quad \frac{\partial\Pi_x}{\partial z} = \frac{k_E^2}{k^2}\frac{\partial\Pi_{Ex}}{\partial z} \quad (20)$$

and, combining all these, gives for the two components:

$$\begin{aligned} \Pi_x &= \frac{k_E^2}{k^2}\Pi_{Ex}, \quad \frac{\partial\Pi_x}{\partial z} = \frac{k_E^2}{k^2}\frac{\partial\Pi_{Ex}}{\partial z} \\ \Pi_z &= \frac{k_E^2}{k^2}\Pi_{Ez}, \quad \frac{\partial\Pi_z}{\partial z} - \frac{\partial\Pi_{Ez}}{\partial z} = \frac{\partial\Pi_{Ex}}{\partial x} - \frac{\partial\Pi_x}{\partial x} \end{aligned} \quad (21)$$

Since the dipole is aligned along the x-direction, we can consider that a cylindrical coordinate axis is also in this direction and the wave equation is similar to (7) with x and z interchanged. The final result for Π_x can be determined in an analogous way to the vertical dipole. Thus we have:

$$F = \frac{\lambda}{\mu}\left(-1 + \frac{2\mu}{\mu + \mu_E}\right), \quad F_E = \frac{k^2}{k_E^2}\frac{2\lambda}{\mu + \mu_E} \tag{22}$$

$$\begin{aligned} \Pi_x &= \frac{e^{jkR}}{R} - \frac{e^{jkR'}}{R'} + 2\int_0^\infty J_0(\lambda r)e^{-\mu(z+h)}\,\frac{\lambda d\lambda}{\mu + \mu_E} \\ \Pi_{Ex} &= 2\frac{k^2}{k_E^2}\int_0^\infty J_0(\lambda r)e^{\mu_E z - \mu h}\,\frac{\lambda d\lambda}{\mu + \mu_E} \end{aligned} \tag{23}$$

where R' is the distance to the image of the dipole in the ground.

For the z-component of the Hertz potential, we set the axis of a cylindrical coordinate system in the z-direction. The wave equation is a Helmholtz equation:

$$(\nabla^2 + k^2)\Pi_z = 0 \tag{24}$$

This has solutions in cylindrical coordinates of the form (see Mathews and Walker [3], or any standard text on differential equations, including [1]):

$$\Pi_z = J_m(\lambda r)e^{-\mu z}e^{jm\phi} \tag{25}$$

From the boundary conditions, Sommerfeld shows that the azimuthal factor is $\cos\phi$; therefore the eigenfunctions now involve J_1 rather than J_0. The primary field does not affect Π_z because it arises only from currents parallel to the x-axis. On the other hand, the currents in the ground are created by the changing electromagnetic field, which induces currents in the x-direction. The radial electric field due to the wire also produces currents. These currents have a component in the z-direction.

Sommerfeld's results are:

$$\begin{aligned} \Pi_z &= -\frac{2}{k^2}\cos\phi\int_0^\infty J_1(\lambda r)e^{-\mu(z+h)}\,\frac{\mu - \mu_E}{\mu k_E^2/k^2 + \mu_E}\lambda^2 d\lambda, \quad z > 0 \\ \Pi_z &= -\frac{2}{k_E^2}\cos\phi\int_0^\infty J_1(\lambda r)e^{\mu_E z - \mu h}\,\frac{\mu - \mu_E}{\mu k_E^2/k^2 + \mu_E}\lambda^2 d\lambda, \quad z < 0 \end{aligned} \tag{26}$$

III. Banos Theory

Banos [2] also treats the Hertzian dipole and uses a time-frequency factor equal to $e^{-j\omega t}$. He notes that, as well as Maxwell's equations, the problem requires the continuity equation:

$$\nabla \cdot \mathbf{J} = j\omega\rho \tag{27}$$

where ρ is the charge density. He introduces the intrinsic impedance, ζ, and admittance, η, of the medium; these are defined by:

$$k\zeta = \omega\mu, \quad k\eta = \omega\varepsilon + j\sigma \tag{28}$$

The inhomogeneous Helmholtz equation for the Hertz potential is shown to be:

$$(\nabla^2 + k^2)\mathbf{\Pi} = -j\mathbf{J}^0 / k\eta \tag{29}$$

where $\mathbf{J}^0$ is the impressed current density. For a horizontal dipole on the x-axis we have:

$$\mathbf{J}^0 = \hat{\mathbf{e}}_x p\delta(x)\delta(y)\delta(z-h) \tag{30}$$

where $p = I\Delta l$ is proportional to the electric dipole moment of the antenna. The boundary conditions in terms of the Hertz potential for the vertical and horizontal dipoles are the same as those of Sommerfeld.

Banos points out that the formalism can be expressed in terms of a triple Fourier transform. Apparent differences reflect the choice of coordinate systems in the spatial domain, the transform domain or both. As we have seen, Sommerfeld's model is based on cylindrical coordinate systems in both the spatial domain and the transform domain.

His treatment focuses on the Green's function, G. for a unit electric dipole moment, this is given by:

$$G = \frac{e^{jkR}}{R}\quad , \tag{31}$$

which is a solution of the inhomogeneous Helmholtz equation:

$$(\nabla^2 + k^2)G = -4\pi\delta(x)\delta(y)\delta(z) \tag{32}$$

In the case of the horizontal dipole, Banos indicates the following relations:

$$\begin{aligned} \mathbf{\Pi} &= \hat{\mathbf{e}}_r\Pi_x\cos\phi - \hat{\mathbf{e}}_\phi\Pi_x\sin\phi + \hat{\mathbf{e}}_z\Pi_z \\ E_r &= \frac{\partial}{\partial r}(\nabla\cdot\mathbf{\Pi}) + k^2\Pi_x\cos\phi \\ E_\phi &= \frac{1}{r}\frac{\partial}{\partial\phi}(\nabla\cdot\mathbf{\Pi}) - k^2\Pi_x\sin\phi \\ E_z &= \frac{\partial}{\partial z}(\nabla\cdot\mathbf{\Pi}) + k^2\Pi_z \\ H_r &= -\frac{jk^2}{\omega\mu_0}\left(\sin\phi\frac{\partial\Pi_x}{\partial z} + \frac{1}{r}\frac{\partial\Pi_z}{\partial\phi}\right) \\ H_\phi &= -\frac{jk^2}{\omega\mu_0}\left(\cos\phi\frac{\partial\Pi_x}{\partial z} - \frac{\partial\Pi_z}{\partial r}\right) \\ H_z &= \frac{jk^2}{\omega\mu_0}\left(\sin\phi\frac{\partial\Pi_x}{\partial r}\right) \end{aligned} \tag{33}$$

Results for the electric and magnetic fields in the air and ground are found in terms of various integrals related to Green's functions for Hertzian dipoles located at different heights in each of the two media. There is also a discussion on choices of branch cuts and the handling of poles in the complex plane; this is useful when explicit results are needed.

Most of Banos's book is devoted to saddle point methods and their utility in deriving results in particular cases. Saddle point methods are instructive in that they offer physical insight into how fields propagate. For example the surface waves of Sommerfeld emerge naturally from this type of treatment. However, it is equally obvious that practical problems must often be sub-divided into various categories, such as the field near the interface and the field near the vertical axis. This renders the treatment of long antennas rather difficult and suggests that numerical methods are much simpler. Therefore saddle point methods will not be pursued here.

IV. WAIT THEORY

Wait provides a useful summary of the basic electromagnetic theory in the appendix of his book [4]. This begins with the usual theory of radiation from an elementary impressed source of current. The Hertz vector is introduced and it is asserted that, in free space, only one component of the Hertz potential is required and this is parallel to the impressed current on the *z*-axis, say. This is reasonably obvious because we know that the magnetic field lines are all perpendicular to the radial and axial vectors and circle the *z*-axis. Only the curl of the *z*-component of the Hertz vector is purely in the azimuthal direction, so that this is all that is required.

It should be noted that, in contrast to the other two authors, Wait employs a time dependence of the form $e^{j\omega t}$.

The concept of magnetic current, **M**, is discussed by exploiting the symmetry between electric and magnetic vector equations, viz:

$$\begin{aligned} \nabla \times \mathbf{E} &= -(j\mu\mu_0\omega\mathbf{H} + \mathbf{M}) \\ \nabla \times \mathbf{H} &= (\sigma + j\omega\varepsilon\varepsilon_0)\mathbf{E} + \mathbf{J} \end{aligned} \tag{34}$$

Here **J** and **M** are impressed currents, though the latter is fictitious. This leads directly to the introduction of the magnetic Hertz vector, **Π***, which is given by:

$$\mathbf{\Pi}^* = \frac{1}{4\pi j\mu\mu_0\omega}\int_V \frac{e^{-jkR}}{R}\mathbf{M}dv \tag{35}$$

Wait then considers the radiation from a small current loop using the previous type of approach with the usual Hertz vector (by integrating around the current loop). He shows that the result can equally be derived using the magnetic Hertz potential:

$$\Pi_z^* = \frac{Kl}{4\pi j\mu\mu_0\omega}\frac{e^{-jkR}}{R} \tag{36}$$

where

$$Kl = j\mu\mu_0\omega IS \tag{37}$$

Here, *K* is the magnetic current of an element of length *l* and *IS* is the magnetic moment of the loop.

He then gives an example of the radiation from a solenoid, which can be represented by a magnetic current of finite length.

Wait also provides insight into the fields of line sources in a homogeneous medium with no interface nearby. However, this is similar to the treatments in other standard texts. He shows that the fields are proportional to modified Bessel functions of the second kind (which are proportional in turn to Hankel functions).

In later papers, Wait shows that the magnetic Hertz vector is a useful tool for representing circulating currents when an infinite line source is parallel to the interface with a conducting medium.

ACKNOWLEDGMENT

This report forms a part of the support for a Defence Research and Development Canada (DRDC Suffield) contract involving the detection of command wires: Contract number W7702-125254/001/EDM.

REFERENCES

[1] A. Sommerfeld, *Partial Differential Equations in Physics*, Volume VI, Academic Press, London, 1964.

[2] A. Banos Jr., *Dipole Radiation in the Presence of a Conducting Half-Space*, Pergamon Press, Oxford, 1966.

[3] J. Mathews and R.L. Walker, *Mathematical Methods of Physics*, 2nd Ed., W.A. Benjamin Inc., New York, 1970.

[4] J.R. Wait, *Electromagnetic Radiation from Cylindrical Structures*, Pergamon Press, New York, 1959.