Ship Wakes in Shallow Waters

James K.E. Tunaley

***Abstract*—The observed opening angle of a surface gravity wave ship wake is often significantly less than that of the classical Kelvin wake and one possibility is that this is due to shallow waters. The purpose of this paper is to describe a simple means of finding the angle as a function of water depth and speed of the ship. It is shown that the wake angle can be greater or less than that of the Kelvin wake and that a narrow wake can be produced only when both the depth is quite small and the ship speed is high. It is unlikely that the observations can be attributed entirely to shallow water. Another possible problem is identified.**

***Index Terms*—Ship Wake, Kelvin Wake, Narrow-V Wake.**

I. INTRODUCTION

THE classical Kelvin wake comprises surface gravity waves and is produced by a ship moving in calm water of infinite depth [1]. The disturbance created by the ship in a reference frame attached to it is stationary and this implies that the wake pattern and wave amplitudes are also stationary in this reference frame. It is well known that the wake has a half angle of about 19.47°. However, wakes that resemble the Kelvin wake but with half angles much less than this are often observed in both optical and radar imagery [2][3]. These have been called "narrow-V" wakes. One possibility is that these wakes can be attributed to shallow water. Another is that small ships moving at high speed only excite waves of small wavelength so that the typical transverse and cusp wave structures of the Kelvin wake are absent [2].

The purpose of this paper is to examine the effect of finite water depth on the wake angle. The method, which resembles that used in other work [4][5][6], is applicable to internal wave wakes as well as surface gravity wave wakes and is very simple.

The second proposal will be discussed in another paper but it is worth noting that the effect of the hull shape on the amplitude factors in the Kelvin wake theory was discussed at length by Havelock in 1934 [7]. He pointed out that the theory implies that the wake of small, broad, shallow draft ships should comprise mainly divergent waves as observed by Admiral Taylor. The case of high speed planing craft is simply an extreme example.

Most features of gravity wave wakes can be treated using the concepts of wave speed and group velocity. Strictly, the approach applies to the far wake but, in practice, it is found that the wake crest patterns are close to reality even near a ship. This can be understood by noting that the wave and group speeds have a simple and well-defined physical interpretation: the group velocity represents the velocity of a wave packet and the phase speed is the speed of a crest within the packet.

The waves at a point within the wake are constrained by their propagation speed and their stationarity in the reference frame of the ship. The first constraint can be expressed in terms of geometrical conditions and the second by a simple phase condition that can be expressed in terms of the phase speed. The latter is equivalent to noting that the component of the ship speed parallel to the phase velocity is equal to the phase speed. A component of the ship velocity perpendicular to the phase velocity just causes crests to slide parallel to themselves and the crest pattern is unaffected.

When the water is very deep, waves traveling at high speed can propagate on the surface. The group speed is typically approximately half the wave speed. In contrast, at large wavelengths in shallow water both group and phase speeds approach the same limiting value. This implies that there are two regimes. When the ship is moving slowly at a speed less than the maximum wave and group speed, transverse waves can propagate at the ship's speed and the wake will somewhat resemble the Kelvin wake. There will be both divergent and transverse wave systems intersecting in a cusp region. When the ship is moving at a speed greater than the maximum, transverse waves cannot propagate and the wake will consist of only divergent waves. In this regime, the wake half angle, $\beta_{\max}$, will be determined by the maximum group velocity, $c_{g\max}$, and the ship speed, U, according to:

$$\sin\beta_{\max} = c_{g\max} / U \tag{1}$$

The dispersion relationship for waves in shallow water permits the phase speed, c, and the group velocity, c_g, to be found. The dispersion relation is given by [1]:

$$\omega^2 = gk\tanh(hk) \ , \tag{2}$$

where ω is the angular frequency, g is the acceleration due to gravity, k is the angular wave number and h is the depth. When the depth is very large, the hyperbolic term can be ignored except when k is small or, in other words, when the wavelength is of the order of the depth or greater. With this exception, the dispersion relation reduces to the familiar deep-water version.

The phase and group speeds (c and c_g) can be derived as a function of k directly from the dispersion relation and, after

---

First published June 4th, 2014: minor modifications June 24th, 2014. James K.E. Tunaley is President of London Research and Development Corporation, Ottawa, Canada. Phone: 613.839.7943; e-mail: jtunaley@London-Research-and-Development.com.

some minor manipulation, we have:

$$c(k) = \frac{\omega}{k} = \sqrt{\frac{g\tanh(hk)}{k}}$$
$$c_g(k) = \frac{d\omega}{dk} = \frac{c}{2}\left(1 + \frac{hk}{\sinh(hk)\cosh(hk)}\right). \quad (3)$$

When hk is very large, the group speed is half the phase speed. When hk is small, the group speed approaches the phase speed.

The phase constraint is simply:

$$c = U\cos\theta, \quad (4)$$

where θ is the angle between the group velocity and the ship velocity vectors (see Fig. 1).

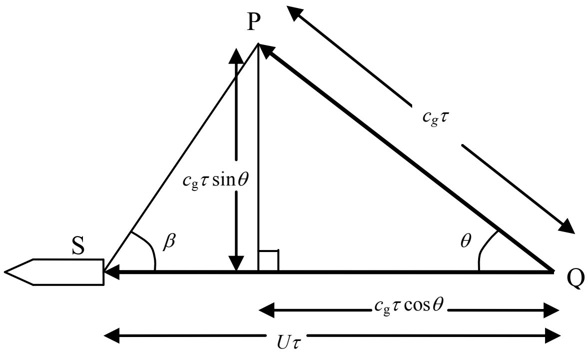

Figure 1. Wake geometry.

II. THEORY

The wake geometry is shown in Fig. 1. The waves that arrive at point P arise from a disturbance at point Q on the ship's track. Here τ is the propagation time. The ship is currently at point S. The geometrical constraint indicates that the waves at P are related to the angle, β, within the wake by:

$$\tan\beta = \frac{c_g\sin\theta}{U - c_g\cos\theta} \quad (5)$$

Application of the phase condition (4) yields an equation that is valid for a stationary wake associated with any reasonable dispersion relation:

$$\tan\beta = \frac{c_g\sqrt{U^2 - c^2}}{(U^2 - cc_g)} \quad (6)$$

Therefore, for each value of k that results in a phase speed less than the ship speed, there will be a corresponding value of β.

In the present context it is convenient to switch to dimensionless variables to remove explicit reference to g and U. Thus we define:

$$C = c/U; C_g = c_g/U$$
$$\Omega = \omega U/g; \kappa = kU^2/g; \quad (7)$$
$$H = hg/U^2$$

The dimensionless phase and group speeds are given by:

$$C(\kappa) = \sqrt{\frac{\tanh(H\kappa)}{\kappa}}$$
$$C_g(\kappa) = \frac{C}{2}\left(1 + \frac{H\kappa}{\sinh(H\kappa)\cosh(H\kappa)}\right), \quad (8)$$

while (6) becomes:

$$\tan\beta = \frac{C_g\sqrt{1 - C^2}}{(1 - CC_g)}. \quad (9)$$

Fig. 2 shows a plot of the half angle, β, against κ for the Kelvin wake ($H = \infty$). As can be seen, only values of β up to a maximum of about 19.47° can occur as expected. Divergent waves correspond to that part of the plot to the right of the maximum and transverse waves to that part to the left of it.

A similar plot is shown in Fig. 3 for $H = 1.2$. Relative to the Kelvin wake plot, the curve is shifted towards the origin and the maximum wake half angle is now increased to about 37°. As H is reduced towards 1, this tendency continues. However, when $H < 1$, the maximum occurs at $\kappa = 0$; the transverse structure disappears and only divergent waves exist. The plots resemble Fig. 4, which is for $H = 0.1$.

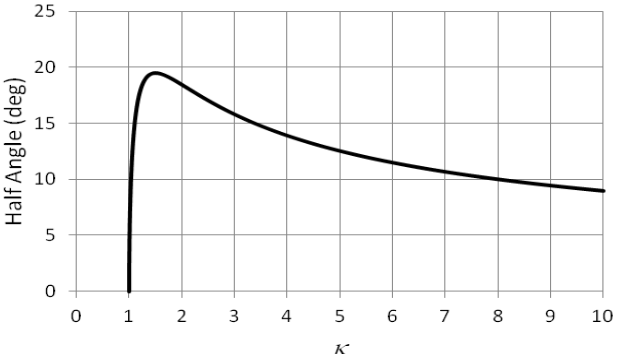

Figure 2. Relation between β and κ for the Kelvin wake.

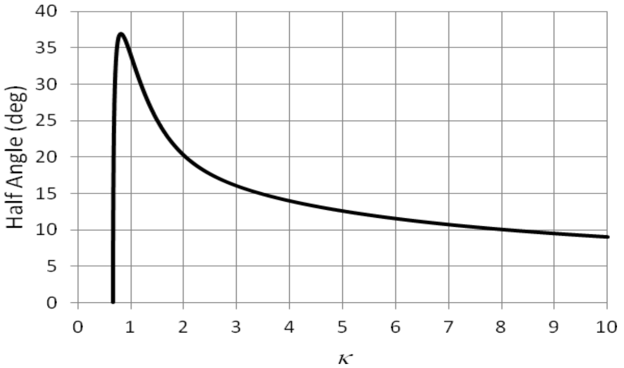

Figure 3. Plot of β against κ for $H = 1.2$.

As already implied, (8) and (9) demonstrate that the plots of β against κ are only a function of the dimensionless depth, H, so that H characterizes the maximum shallow water wake angle. This is plotted in Fig. 5. From (8) and (9), the maximum phase and group speeds occur when $\kappa \to 0$, i.e. $C_g \to C \to H^{1/2}$. The condition for the existence of transverse waves is that the maximum phase speed is greater than the ship speed or $C_{\max} > 1$. Therefore a hiatus is expected at $H = 1$. When $H < 1$, only divergent waves can exist but when $H > 1$ there can be both divergent and transverse waves. This hiatus can be seen in Fig. 5. In very shallow waters with high ship speeds the wake is very narrow. As the water becomes deeper or the ship speed is reduced, the wake broadens and becomes wider than that of the classical Kelvin wake. The half angle eventually reaches 90º. Further increase in depth or reduction of ship speed causes the wake angle to decrease rapidly when it approaches that of the classical Kelvin wake.

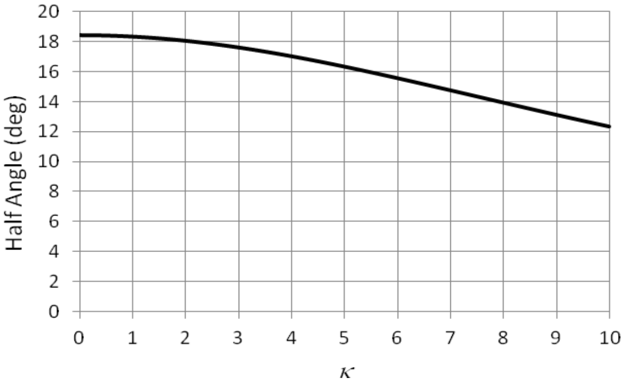

Figure 4. Plot of *β* against *κ* for *H* = 0.1.

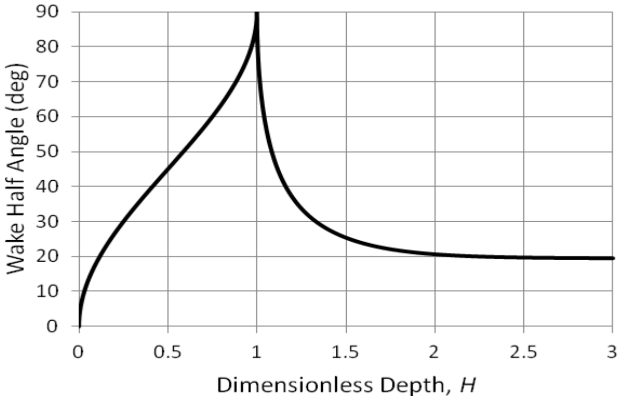

Figure 5. The maximum wake half angle versus dimensionless depth.

III. DISCUSSION

Fang et al. [6] present their results in terms of a Froude number, *Fr*, given by:

$$Fr = U / \sqrt{gh} = H^{-1/2}. \tag{10}$$

The present results are provided in Figs. 6 and 7 in the same way as in [4] and [6]. They seem to be similar to those in [4] but Havelock's figure is not very accurate. Though there is qualitative agreement with [6], there are serious differences and the reader is advised to verify any calculations independently.

It must be emphasized that shallow water wakes can be broader than the classical Kelvin wake as well as narrower. To appreciate the practical effects, Fig. 8 shows the maximum wake angle as a function of ship speed for two depths of water (2 m and 10 m).

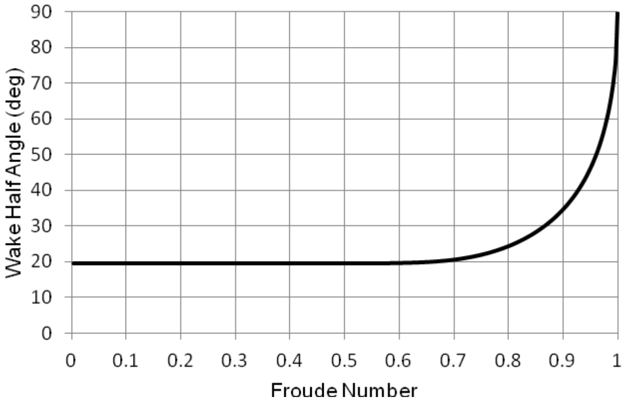

Figure 6. Wake half angle as a function of Froude number.

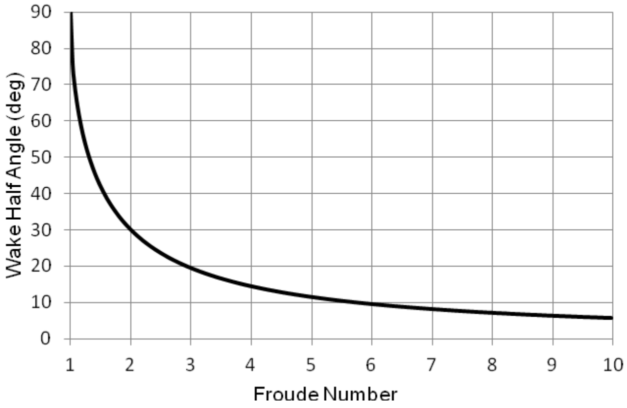

Figure 7. Wake half angle as a function of Froude number.

The theory suggests that narrow-V wakes associated with shallow water would not normally be observed in waters deeper than 10 m because the ship speed would have to be too high. However, go-fast boats as well as typical power boats that are used for pleasure can certainly satisfy the conditions for narrow-V wakes in moderately shallow waters. The theory also suggests that wide wakes should be quite common.

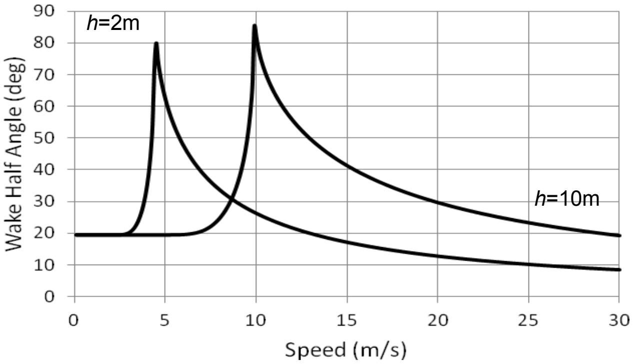

Figure 8. Maximum half angle as a function of ship speed for depths of 2 m and 10 m.

IV. CONCLUSION

A simple calculation of shallow water wake angles has been described and this is also applicable to other types of wake including the wake generated on an internal layer.

A narrow-V wake is only produced when the water is quite shallow and the ship speed is very high. Narrow-V wakes could be produced by planing craft in practice but traveling at high speed in shallow water cannot be recommended. At realistic speeds, the wake width should usually be equal to or greater than that of the Kelvin wake. Large naval and commercial vessels with a draft of a few meters or more do not operate in shallow water especially at high speed. Therefore narrow-V wakes are not expected for these ships.

The wakes from planing craft as well as large displacement ships moving at practical speeds should often exhibit wakes that are significantly wider than the Kelvin wake. This does not appear to be consistent with observations. Wide wakes have sometimes been observed by the author in space-borne synthetic aperture radar images of the ocean using the Radon transform. However, the principal wake components are usually confined to the Kelvin angle or less as in the optical wakes reported in [2].

This problem of the lack of observations of wide wakes in shallow water needs to be resolved by a full simulation of wake amplitudes.

ACKNOWLEDGMENT

Thanks are due to Germain Rousseaux (CNRS) for his helpful comments especially in pointing out the original work of Havelock in 1908.

REFERENCES

[1] J. Lighthill, *Waves in Fluids*, Cambridge Univ. Press, Cambridge, England, 1978.
[2] M. Rabaud and F. Moisy, “Ship Wakes: Kelvin or Mach Angle”, Phys. Rev. Lett, DOI: PRL 110, 215503(2013), March 2013.
[3] K. Case, C. Callan, R. Dashen, R. Davis, W.Munk, J.Vesecky, K. Watson, F. Zachariasen, SEASAT III and IV, Report JSR-84-203, JASON**,** MITRE Corporation, 1820 Dolley Madison Boulevard, McLean, Virginia 22102, August 1984.
[4] T.H. Havelock, “The Propagation of Groups of Waves in Dispersive Media, with Application to Waves on Water Produced by a Travelling Disturbance”, Proc. R. Soc. London A, Vol. 81, DOI: 10.1098/rspa.1908.0097, 1908.
[5] J.K.E. Tunaley. “The Unsteady Wake from a Body Moving Near an Internal Layer”, Proc. 5th Canadian Conference on Marine Hydrodynamics, St. Johns, Newfoundland, 1999.
[6] M.C. Fang, R.Y. Yang and I.V. Shugan, “Kelvin Ship Wake in the Wind Waves Field and on the Finite Sea Depth”, J. Mech., Vol. 27(1), pp 71-77, March 2011.
[7] T.H. Havelock, “Wave Patterns and Wave Resistance”, Trans. RINA, DOI: 10.3940/rina.trans.1934.27, 1934.