K-DISTRIBUTION ALGORITHM

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Abstract—**Approximation is found for the K-distribution probability density function. This avoids detailed calculation of modified Bessel functions of the second kind. The K-distribution is often used to model sea clutter in radar images and a simplified algorithm is important for the automatic detection of ships in radar images. The approximation is sufficiently accurate for this application and could significantly reduce the time required to process sea clutter. The methodology contributes to an understanding of why the K-distribution is successful in applications to target detection.**

Index Terms— **K-distribution, detection threshold, sea clutter, ship detection, CFAR, SAR.**

I. Introduction

THE K-distribution often describes the intensity statistics of radar sea clutter and can be used in the automatic detection of ships from satellite radar imagery. The technique involves the specification of a Constant False Alarm Rate (CFAR), which implies that a threshold of detection is set according to the local statistics of the clutter at each point in the image plane.

The K-distribution arises quite naturally because each area corresponding to a radar resolution cell in range and azimuth usually contains a large number of independent scatterers. In the complex domain appropriate to the radar I and Q signals, the sum of identically distributed random scatterers leads to two dimensional normal statistics [1]. The squared magnitude of the signal, which is the intensity, turns out to be exponentially distributed. The scatterers themselves can be considered as independent Bragg waves within a small region of wave-vector space corresponding to the radar resolution. If the mean squared amplitude of these waves varies across the sea surface, for example as would be expected from variations in the wind speed, then the mean of the exponential distribution must be randomized. When the randomizing probability density is assumed to be gamma distributed, the result is K-distributed clutter [2]; K refers to a modified Bessel function of the second kind.

The images from a Synthetic Aperture Radar (SAR) are often averaged, essentially in time. These are called multi-look images. The mean intensity within a single resolution cell is not likely to change over a time of a fraction of a second but the phasing of the Bragg waves will be almost independent from look to look. Therefore the multi-looking operation converts the exponential distribution to a gamma distribution of order L, where L is the number of independent looks (which is usually less than the actual number of looks) [1]. When this is randomized using a similar gamma distribution to the single look case, the result is compound K-distributed clutter [3], [4].

The order of the randomizing gamma distribution ν is determined empirically. When ν is very large, the randomizing distribution is highly concentrated around its mean and the effect of randomization is negligible; the overall distribution is gamma of order L. When ν is small, the randomizing distribution is broad and large values of the intensity are more likely to occur, which increases the probability of a false alarm. Experiments and the analysis of sea clutter suggest that the mean intensity within each resolution cell is indeed close to gamma distributed but the theoretical justification for this is weak. However, it is plausible that many bell-shaped distributions close to gamma would suffice and yield a distribution for randomized clutter that would be close to compound K-distributed; this will be confirmed.

The calculation of an appropriate CFAR threshold for each region of an image can be implemented by determining the value of L from the radar and processing specifications and a value of ν from the clutter data itself. The value of ν can be estimated from the mean and variance of a clutter cell. The threshold is determined from a numerical calculation of the modified Bessel function and the subsequent evaluation of the probability distribution. Unfortunately the last of these operations can be a time consuming process. Because the desired false alarm rates are typically very small ($\ll 10^{-6}$ per resolution cell), the process can be accelerated greatly by using an asymptotic approach.

II. Theory

Two cases are considered. The first case is that of single look imagery, i.e. $L = 1$. Here both the basic probability density function and the distribution are exponential; since the distribution can be expressed in closed form, a direct evaluation is possible. In the second case (where $L > 1$), the basic probability density is gamma. However, this cannot be integrated in closed form. Nevertheless, it is possible to find the randomized probability density function and then to integrate this to determine the overall probability distribution and from this the threshold. The last operation parallels the usual numerical approach.

For single look imagery, the randomized distribution function for the intensity, X, with unit mean is given by:

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$$P\{X > x\} = \int_0^\infty e^{-x/z} \frac{\nu}{\Gamma(\nu)} (\nu z)^{\nu-1} e^{-\nu z} dz \quad (1)$$

where Γ is the usual gamma function related to the factorial.

This can be integrated to yield a K-distribution but, to find an asymptotic form for large x, it is re-cast into the following form:

$$P\{X > x\} = \frac{\nu}{\Gamma(\nu)} \int_0^\infty \exp\left(-x/z - \nu z + (\nu-1)\ln(\nu z)\right) dz \quad (2)$$

The argument of the exponential is now written:

$$f = -\frac{x}{z} - \nu z + (\nu-1)\ln(\nu z) \quad (3)$$

The integrand peaks sharply when f is a maximum or when $df/dz = 0$:

$$\frac{df}{dz} = f' = \frac{x}{z^2} - \nu + \frac{\nu-1}{z} = 0 \quad (4)$$

This occurs when:

$$z_0 = \frac{\nu-1}{2\nu}\left(1 \pm \sqrt{1 + \frac{4\nu x}{(\nu-1)^2}}\right) \quad (5)$$

The positive sign is used for $\nu > 1.0$ and the negative sign for $\nu < 1.0$. The second derivative of f is:

$$\frac{d^2 f}{dz^2} = f'' = -\frac{2x}{z_0^3} - \frac{\nu-1}{z_0^2} \quad (6)$$

For the values of ν that are typically encountered, the integrand is very concentrated about exp(f) evaluated at z_0. Thus we have:

$$\begin{aligned} P\{X > x\} &\rightarrow \frac{\nu}{\Gamma(\nu)} \int_0^\infty \exp\left(f(x, z_0) + \frac{f''(x, z_0)(z - z_0)^2}{2} \right) dz \\ &= \frac{\nu\sqrt{2\pi}}{\Gamma(\nu)} \frac{\exp(f(x, z_0))}{\sqrt{|f''(x, z_0)|}} \end{aligned} \quad (7)$$

There are two possible problems. The first is when $\nu = 1$. This is resolved by noting that $z_0 = x^{1/2}$. The second problem is when $\nu < 1$ and the typical gamma function algorithm may not be accurate. To overcome this, the following can be employed:

$$\Gamma(\nu) = \frac{\Gamma(\nu+1)}{\nu} \quad (8)$$

When $L > 1$, the randomized density for the intensity is given by:

$$p(x) = \int_0^\infty \frac{L}{z\Gamma(L)} \left(\frac{Lx}{z}\right)^{L-1} e^{-Lx/z} \frac{\nu}{\Gamma(\nu)} (\nu z)^{\nu-1} e^{-\nu z} dz \quad (9)$$

Using the same approach as before we have:

$$\begin{aligned} f &= -Lx/z - \nu z + (L-1)\ln(Lx/z) + (\nu-1)\ln(\nu z) \\ f' &= Lx/z^2 - \nu + (\nu - L)/z \\ f'' &= -2Lx/z^3 - (\nu - L)/z^2 \end{aligned} \quad (10)$$

Thus z_0 is given by:

$$z_0 = \frac{\nu - L}{2\nu}\left(1 \pm \sqrt{1 + \frac{4L\nu x}{(\nu - L)^2}}\right) \quad (11)$$

The positive sign is used when $\nu > L$ and the negative sign when $\nu < L$; when $\nu = L$, then $z_0 = x^{1/2}$. The final result is

$$p(x) = \frac{\nu L\sqrt{2\pi}}{z_0 \Gamma(\nu)\Gamma(L)} \frac{\exp(f(x, z_0))}{\sqrt{|f''(x, z_0)|}} \quad (12)$$

III. RESULTS

In practice a Probability of False Alarm (PFA) is specified and the threshold of detection is calculated as a multiple of the mean. A comparison was made between an accurate calculation of the thresholds (using the K-distribution based on modified Bessel functions of the second kind; see Appendix) with approximate results for PFAs of 10^{-9} and 10^{-6}. The calculation required an evaluation of the probability distributions, which for $L = 4$ were found by numerical integration of the probability density functions. Some of the results are shown in Table I. With a PFA of 10^{-9} and when $\nu > 0.1$ and $1 \le L \le 100$, it has been verified that the accuracy of the approximation is better than 0.1%.

TABLE I
THRESHOLDS OF DETECTION

		$L = 1$		$L = 4$	
PFA	ν	Accurate	Approx.	Accurate	Approx.
10^{-9}	0.5	214.7	214.8	91.59	91.62
10^{-9}	5.0	47.49	47.50	18.796	18.800
10^{-9}	50.0	24.24	24.24	8.841	8.842
10^{-6}	0.5	95.43	95.55	46.40	46.43
10^{-6}	5.0	25.69	25.70	11.263	11.267
10^{-6}	50.0	15.337	15.338	6.128	6.128

IV. CONCLUSIONS

Approximate forms of the K probability distribution ($L = 1$) and for the probability density function ($L > 1$) have been found. These appear to be sufficiently accurate for estimating the thresholds for the detection of ships over the range of sea clutter parameters likely to be encountered.

The calculation is much simpler than a detailed calculation involving modified Bessel functions of the second kind and, for realistic PFAs, is accurate to within better than 0.1%. In C++ or C# and apart from the gamma function, the computer code for the probability density needs no loops and comprises less than 18 lines. Therefore the approximation should be useful in reducing the time required to process SAR image data for automatic ship detection.

The methodology involves replacement of the integrand by an approximation that includes a generic Gaussian component. Because the thresholds for the approximate calculation are very close to those derived by an accurate evaluation of the K-distribution, the details of the randomizing distribution are not critical and a variety of smooth, well-behaved randomizing distributions will give similar results. This helps to explain why the K-distributions are so useful in modeling clutter.

APPENDIX

The integrals involved in the K probability density and distribution can be evaluated using a relation from [5]:

$$K_\nu(x) = \int_0^\infty e^{-x\cosh\theta} \cosh(\nu\theta)\, d\theta, \quad R(x) > 0 \tag{13}$$

For the case of the distribution with $L = 1$, we have:

$$\begin{aligned} P\{X > x\} &= \int_0^\infty e^{-x/z} \frac{\nu}{\Gamma(\nu)} (\nu z)^{\nu-1} e^{-\nu z} dz \\ &= \frac{\nu}{\Gamma(\nu)} \int_0^\infty e^{-x/z-\nu z} (\nu z)^{\nu-1} dz \end{aligned} \tag{14}$$

This can be converted into the form of (13) using the substitution:

$$z = \sqrt{\frac{x}{\nu}} e^\theta \tag{15}$$

i.e.

$$\begin{aligned} P\{X > x\} &= \frac{\nu^\nu}{\Gamma(\nu)} \int_0^\infty \exp(-2\sqrt{\nu x} \cosh\theta) e^{\nu\theta} d\theta \\ &= \frac{2\,(\nu x)^{\nu/2}}{\Gamma(\nu)} K_\nu(2\sqrt{\nu x}) \end{aligned} \tag{16}$$

It can easily be verified that this tends to one as x goes to zero and to zero as x goes to infinity.

For the case of multi-look imagery, we have for the probability density:

$$p(x) = \int_0^\infty \frac{L}{z\Gamma(L)} \left(\frac{Lx}{z}\right)^{L-1} e^{-Lx/z} \frac{\nu}{\Gamma(\nu)} (\nu z)^{\nu-1} e^{-\nu z} dz \tag{17}$$

By similar reasoning it can be shown that:

$$p(x) = \frac{2(L\nu x)^{(L+\nu)/2}}{x\Gamma(L)\Gamma(\nu)} K_{\nu-L}(2\sqrt{L\nu x}) \tag{18}$$

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